1.0

1.1

1.25

4.5
5.0
5.5

2.8

3.2

3.6

4.0

2.5

2.2

2.0

1.8

1.4   1.6

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CONTRACT N00014-75-C-0196

TECHNICAL REPORT NO. 37

ON THE FAILURE OF ELLIPTICITY AND THE EMERGENCE OF DISCONTINUOUS DEFORMATION GRADIENTS IN PLANE FINITE ELASTOSTATICS

BY

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AUG 5 1977

DIVISION OF ENGINEERING AND APPLIED SCIENCE

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA

JUNE 1977

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Summary

This investigation concerns equilibrium fields with discontinuous displacement gradients, but continuous displacements, in the theory of finite plane deformations of possibly anisotropic, compressible elastic solids. "Elastostatic shocks" of this kind, which resemble in many respects gas-dynamical shocks associated with steady flows, are shown to exist only if and when the governing field equations of equilibrium suffer a loss of ellipticity. The local structure of such shocks, near a point on the shockline, is studied with particular attention to weak shocks, and an example pertaining to a shock of finite strength is explored in detail. Also, necessary and sufficient conditions for the "dissipativity" of time-dependent equilibrium shocks are established. Finally, the relevance of the analysis carried out here to localized shear failures — such as those involved in the formation of Lüders bands — is discussed.

Introduction

Several years ago — in connection with asymptotic studies of

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crack problems within finite elasticity theory — we encountered the fact
that the field equations of nonlinear elastostatics may suffer a loss of ellip-
ticity for some elastic materials in the presence of sufficiently severe lo-
cal deformations. A special isotropic material of this kind was explored
in [1], where we determined all three-dimensional homogeneous deforma-
tions at which such a breakdown of ellipticity occurs. More recently [2],
we deduced explicit necessary and sufficient conditions, in terms of re-
strictions upon the principal stretches and upon the dependence of the
strain-energy density on the principal stretches, for ordinary and strong
ellipticity of the equations governing finite plane deformations of general
homogeneous and isotropic, compressible hyperelastic solids.

A failure of ellipticity of the elastostatic equations appropriate to
such solids suggests the possible emergence of solution fields exhibiting a
loss of smoothness. In the present paper our interest is directed in par-
ticular at the possibility of solutions that possess finite jump discontinu-
ities in the first displacement gradients across certain curves, while the
displacements themselves still remain everywhere continuous. We call
solutions of this kind "elastostatic shocks" in allusion to gas-dynamical
shocks associated with stationary inviscid flows, to which they bear a
more than casual resemblance. Moreover, since our current concern is
with the local state of affairs near an interior point on a shock-line (line of
displacement-gradient discontinuity), it will be sufficient to confine our at-
tention here to piecewise homogeneous elastostatic shocks: thus we assume
the shock-line to be a straight line, on either side of which the plane defor-
mation under consideration is homogeneous.

Before we proceed to outline the content of this paper, we should

make clear that our motivation in studying elastostatic shocks is physical in origin. Discontinuities of the kind described above arise typically in the idealization of localized shear failures, such as those involved in the formation of Lüders bands.

In Section 1 we recall some prerequisites from the nonlinear theory of plane elastostatic deformations, including the relevant notions of ellipticity. Here we also cite the explicit conditions of ordinary and strong ellipticity established in [2] for the special case of material isotropy, as well as results derived there regarding the inclination of the characteristic curves that accompany a loss of ordinary ellipticity.

Section 2 contains a detailed discussion of response properties in plane strain of two special isotropic elastic materials, which are used in the remainder of the paper to illustrate various general conclusions. The first of these materials, which was also the object of [1], is governed by a stored-energy function originally proposed by Blatz and Ko [3] on the basis of experiments with a highly deformable rubberlike material. The second particular material introduced in Section 2 is strictly hypothetical in nature and is employed later on to bring into evidence certain constitutionally determined qualitative differences in behavior.

A complete definition of piecewise homogeneous elastostatic shocks is spelled out in Section 3, where their kinematics is analyzed in detail. Further, we prove here that the existence of such an equilibrium shock is contingent upon a loss of strong ellipticity of the displacement equations of equilibrium, appropriate to the material at hand, at some homogeneous deformation. The results obtained in Section 3 — like most subsequent results — are not restricted to isotropic materials.

The considerations carried out in Section 3 suggest that if there exists any piecewise homogeneous shock corresponding to a pre-assigned, deformation on one side of the shock-line, there is a one-parameter family of such shocks. In Section 4 we take for granted the existence of a smooth family of shocks in a neighborhood of the shockless state. We then deduce various "weak-shock" results — encompassing the jumps across the shock-line of physically significant field quantities — to dominant order in a shock-strength parameter that measures the departure from the shockless solution. One of the conclusions reached in this manner is that the existence of the presupposed family of equilibrium shocks demands a loss of ordinary ellipticity at the given homogeneous deformation. Moreover, the limiting shock-line at zero shock-strength is found to be a characteristic line associated with this deformation. It should be mentioned that Rudnicki and Rice [4][1] had previously arrived at the required failure of ellipticity in dealing with weak shocks of a related type for a broader class of materials that includes the elastic solid.

Section 5 is devoted to an instructive example of a global analysis of a piecewise homogeneous equilibrium shock of finite strength, based on the first of the two special isotropic elastic materials discussed in Section 2. In this instance explicit results for all possible shocks can be obtained in a transparent elementary form. In particular we find here that every shock admitted is accompanied by a breakdown of ordinary ellipticity at least on one side of the shock-line, regardless of the strength of the shock.

In gas dynamics the sign of the shock strength is determined by the

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[1]See also Rice's [5] more recent paper.

entropy inequality, which models the dissipative character of the process of shock formation in the absence of viscosity. Guided by this fact we generalize in Section 6 the results of Section 3 to time-dependent piece-wise homogeneous quasi-static shocks and then establish an energy identity that serves as a basis for a proposed criterion of dissipativity appropriate to such shocks. Thereafter we obtain necessary and sufficient dissipativity conditions, which lead to an inequality analogous to the entropy condition of gas-dynamics. The dissipation inequality is applied at the end of Section 6 to determine the sign of the shock strength in the special global solution arrived at in Section 5, as well as for weak shocks in the second of the two particular isotropic materials mentioned earlier. This sign, in turn, governs the sign of the jump in the mass density as the shock-line is traversed. We find that the shock strength may be positive or negative depending on the particular nature of the elastic material experiencing the shock.

In Section 7 we combine the special results of Section 5 and the general conclusions reached in Section 6 in an attempt to illustrate the relevance of elastostatic shocks to the phenomenon of Lüders bands in a slab under uni-axial tension or compression. For this purpose we view the evolution of such a band — in the vicinity of a point on the interface between band and slab — as a bifurcation from a homogeneous deformation that has become dynamically unstable following a loss of ellipticity, into a time-dependent piecewise homogeneous equilibrium shock. The ensuing results, though tied to a very limited elastic material, display various striking qualitative features that are reminiscent of familiar experimental observations. In this connection we refer to the work of Hill and

Hutchinson [6], where various bifurcations from a state of plane-strain uni-axial tension are investigated for a class of incompressible, incrementally linear time-independent materials. Among these bifurcations are localized shearing modes.

1. Preliminaries on finite elastostatic plane strain.

In this expository section we recall from [2][1] certain results pertaining to the nonlinear theory of plane elastostatic deformations which are essential to the analysis of the class of discontinuous plane elastostatic fields that constitutes our main objective.

As for notation, we shall use boldface letters to denote vectors and second-order tensors in two dimensions, as well as two-rowed column and square matrices. Further, the same boldface letter will be employed to designate a vector or tensor and its matrix of scalar components in the underlying rectangular cartesian coordinate frame.

Let $\mathcal{R}$ be the open region of the (x_1, x_2)-plane occupied by the interior of the middle cross-section of a cylindrical or prismatic body in its undeformed configuration. A plane deformation of such a body — parallel to the (x_1, x_2)-plane — is described by a suitably smooth and invertible transformation

$$\mathbf{y} = \mathbf{y}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in } \mathcal{R}, \tag{1.1}$$

which maps $\mathcal{R}$ onto a domain $\mathcal{R}_*$ of the same plane. Here $\mathbf{x}$ is the position vector of a generic point in $\mathcal{R}$, $\mathbf{y}(\mathbf{x})$ is its deformation image in $\mathcal{R}_*$, while $\mathbf{u}$ is the displacement vector field. Accordingly, x_α and y_α are the cartesian

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[1] Parts of this resumé are taken verbatim from Section 1 of [2].

material and spatial coordinates, respectively.[1] We call $\underset{\sim}{F}$ the deformation-gradient tensor belonging to the mapping (1.1) and J its Jacobian determinant (area ratio). Thus,

$$\underset{\sim}{F} = \nabla \underset{\sim}{\chi} = [\partial y_\alpha / \partial x_\beta] \ , \quad J = \det \underset{\sim}{F} > 0 \text{ on } \mathcal{R}. \qquad (1.2)[2]$$

Let $\underset{\sim}{\tau}$ be the in-plane Cauchy stress-tensor field accompanying the deformation at hand, so that $\tau_{\alpha\beta}$ stands for the prevailing components of "actual" or "true" stress. The appropriate two-dimensional stress equations of equilibrium, in the absence of body forces, then take the form

$$\text{div} \underset{\sim}{\tau} = \underset{\sim}{0} \ , \quad \underset{\sim}{\tau} = \underset{\sim}{\tau}^T \text{ or } \partial \tau_{\alpha\beta} / \partial y_\beta = 0 \ , \quad \tau_{\alpha\beta} = \tau_{\beta\alpha} \text{ on } \mathcal{R}_*. \qquad (1.3)[3]$$

Next, suppose $\underset{\sim}{\sigma}$ represents the in-plane Piola stress-tensor field corresponding to $\underset{\sim}{\tau}$, whence

$$\left. \begin{array}{l} \underset{\sim}{\sigma} = J \underset{\sim}{\tau} (\underset{\sim}{F}^{-1})^T \text{ or } \sigma_{\alpha\beta} = J \tau_{\alpha\rho} F_{\beta\rho}^{-1} \\[3mm] \underset{\sim}{\tau} = \frac{1}{J} \underset{\sim}{\sigma} \underset{\sim}{F}^T \text{ or } \tau_{\alpha\beta} = \frac{1}{J} \sigma_{\alpha\rho} F_{\beta\rho}, \end{array} \right\} \qquad (1.4)$$

where $\sigma_{\alpha\beta}$ are the components of "nominal" or "pseudo-stress" and $\underset{\sim}{F}^{-1}$, with the components $F_{\rho\beta}^{-1}$, designates the inverse of the nonsingular tensor $\underset{\sim}{F}$. Equations (1.2), (1.3), (1.4) lead to the equilibrium conditions

$$\text{div} \underset{\sim}{\sigma} = \underset{\sim}{0} \text{ or } \partial \sigma_{\alpha\beta} / \partial x_\beta = 0 \text{ on } \mathcal{R}, \qquad (1.5)$$

but $\underset{\sim}{\sigma}$ is in general not symmetric.

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[1] Greek subscripts have the range (1, 2) and we shall employ the usual summation convention.

[2] If $\underset{\sim}{M}$ is a two-by-two matrix with elements $M_{\alpha\beta}$, we alternatively write $[M_{\alpha\beta}]$ in place of $\underset{\sim}{M}$.

[3] A superscript T will always indicate transposition.

Turning now to the governing constitutive relations, we assume that the body is <u>elastic</u> and possesses an <u>elastic potential</u> W, which represents the strain-energy density per unit undeformed volume. Moreover, we restrict the material to be <u>homogeneous</u>. Consequently, W depends upon position on $\mathcal{R}$ exclusively through the deformation-gradient tensor $\underset{\sim}{F}$, and the constitutive law — as far as the in-plane Piola stresses are concerned — becomes

$$\underset{\sim}{\sigma} = W_{\underset{\sim}{F}} \quad \text{or} \quad \sigma_{\alpha\beta} = \partial W / \partial F_{\alpha\beta}. \tag{1.6}$$

Substituting from (1.6) into (1.5), and invoking (1.1), (1.2), one arrives at the displacement equations of equilibrium

$$c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) u_{\gamma,\delta\beta} = 0 \text{ on } \mathcal{R}, \tag{1.7}$$ [1]

provided

$$c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) = \partial^2 W / \partial F_{\gamma\delta} \partial F_{\alpha\beta}. \tag{1.8}$$

For all unit-vectors $\underset{\sim}{N}$, let $\underset{\sim}{Q}(\underset{\sim}{N};\underset{\sim}{F})$ be the symmetric tensor defined by

$$Q_{\alpha\gamma}(\underset{\sim}{N};\underset{\sim}{F}) = c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) N_\beta N_\delta. \tag{1.9}$$ [2]

The quasi-linear system of partial differential equations (1.7) is <u>elliptic</u> at a solution $\underset{\sim}{u}$ (with continuous first and piecewise continuous second partial derivatives on $\mathcal{R}$) and at a material point $\underset{\sim}{x}$, provided

$$\det \underset{\sim}{Q}(\underset{\sim}{N};\underset{\sim}{F}(\underset{\sim}{x})) \neq 0 \text{ for all } \underset{\sim}{N} \text{ with } |\underset{\sim}{N}| = 1, \tag{1.10}$$

where $\underset{\sim}{F}(\underset{\sim}{x})$ is the value at $\underset{\sim}{x}$ of the deformation gradient field generated by $\underset{\sim}{u}$.

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[1] Subscripts preceded by a comma indicate partial differentiation with respect to the corresponding material cartesian coordinates.

[2] The symmetry of $\underset{\sim}{Q}(\underset{\sim}{N};\underset{\sim}{F})$ follows from $c_{\alpha\beta\gamma\delta} = c_{\gamma\delta\alpha\beta}$.

If (1.10) fails to hold, so that

$$\det \underset{\sim}{Q}(\underset{\sim}{N}; \underset{\sim}{F}(\underset{\sim}{x})) = 0 \text{ for some } \underset{\sim}{N} \text{ with } |\underset{\sim}{N}| = 1, \tag{1.11}$$

then $\underset{\sim}{N}$ is normal to a <u>characteristic curve</u> (referred to the undeformed state) associated with the solution $\underset{\sim}{u}$ at $\underset{\sim}{x}$. The characteristic curves, in turn, are the only possible carriers of "weak discontinuities" of $\underset{\sim}{u}$: across such a curve the second normal derivative of $\underset{\sim}{u}$ may exhibit a finite jump discontinuity, whereas its second tangential derivative, as well as the mixed tangential-normal derivative are bound to remain continuous. Thus, the ellipticity condition (1.10) precludes the existence of real characteristics; (1.10) is necessary and sufficient in order that every solution $\underset{\sim}{u}$ of the presupposed smoothness be free of weak discontinuities and hence in fact twice continuously differentiable at the material point under consideration.

Following common usage, we call the system (1.7) <u>strongly elliptic</u> at a solution $\underset{\sim}{u}$ and a material point $\underset{\sim}{x}$, provided $\underset{\sim}{Q}(\underset{\sim}{N}; \underset{\sim}{F}(\underset{\sim}{x}))$ is positive-definite for every unit-vector $\underset{\sim}{N}$, i.e. provided

$$\underset{\sim}{M} \cdot \underset{\sim}{Q}(\underset{\sim}{N}; \underset{\sim}{F}(\underset{\sim}{x})) \underset{\sim}{M} > 0 \tag{1.12}$$

for all unit-vectors $\underset{\sim}{M}$ and $\underset{\sim}{N}$. Clearly, the strong ellipticity of (1.7) at $\underset{\sim}{u}$ and $\underset{\sim}{x}$ implies its ordinary ellipticity.

We proceed next to the particular case of an <u>isotropic</u> body undergoing a plane deformation of the form (1.1). To this end let $\underset{\sim}{C}$ and $\underset{\sim}{G}$, respectively, be the right and the left Cauchy-Green deformation tensors associated with the mapping (1.1), whence

$$\underset{\sim}{C} = \underset{\sim}{F}^{T} \underset{\sim}{F} , \quad \underset{\sim}{G} = \underset{\sim}{F} \underset{\sim}{F}^{T}. \tag{1.13}$$

These two symmetric, positive-definite tensors have the same fundamental scalar invariants and hence common principal values, which are the squares of the local principal stretches (length-ratios); the latter will be denoted by $\lambda_\alpha > 0$. In view of (1.13) and the second of (1.2), the deformation invariants just mentioned obey

$$
\left.
\begin{aligned}
I &= \mathrm{tr}\, \underset{\sim}{C} = \mathrm{tr}\, \underset{\sim}{G} = F_{\alpha\beta} F_{\alpha\beta} = \lambda_1^2 + \lambda_2^2, \\[2mm]
J &= \sqrt{\det \underset{\sim}{C}} = \sqrt{\det \underset{\sim}{G}} = \det \underset{\sim}{F} = \tfrac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\gamma\delta} F_{\gamma\alpha} F_{\delta\beta} = \lambda_1 \lambda_2,
\end{aligned}
\right\}
\quad (1.14)^1
$$

in which $\epsilon_{\alpha\beta}$ stands for the components of the two-dimensional alternator.

In the special instance of an isotropic material subjected to a plane deformation (1.1), the strain-energy density W involves $\underset{\sim}{F}$ merely through the two deformation invariants I and J. In these circumstances one has

$$
W(\underset{\sim}{F}) = W(I, J) = W(\lambda_1, \lambda_2), \tag{1.15}
$$

where — in order to avoid unduly cumbersome notation — we have employed the same functional symbol in three distinct connotations. In particular, $W(I, J)$ and $W(\lambda_1, \lambda_2)$ stand for the plane-strain elastic potential[2] of the material at hand in terms of the deformation invariants and the principal stretches, respectively. From (1.14) one has

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[1] For convenience we use J, rather than the common determinant of $\underset{\sim}{C}$ and $\underset{\sim}{G}$, as the second invariant.

[2] These two alternative forms of the potential are the restrictions to plane deformations of their spatial counterparts, which in general depend upon the three invariants of the three-dimensional deformation tensors and on the triplet of principal stretches. While $W(I, J)$ fully characterizes the in-plane response to a plane deformation, it is insufficient for the determination of the out-of-plane stresses so induced.

$$\frac{\partial I}{\partial F_{\alpha\beta}} = 2F_{\alpha\beta} \ , \ \frac{\partial J}{\partial F_{\alpha\beta}} = \epsilon_{\beta\gamma}\epsilon_{\alpha\delta}F_{\delta\gamma} = JF_{\beta\alpha}^{-1}, \tag{1.16}$$

and (1.6), (1.15), (1.16) furnish the stress-deformation relations

$$\underset{\sim}{\sigma} = W_{\underset{\sim}{F}} = 2W_I \underset{\sim}{F} + JW_J(\underset{\sim}{F}^{-1})^T \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \tag{1.17}$$

or $\quad \sigma_{\alpha\beta} = \dfrac{\partial W}{\partial F_{\alpha\beta}} = 2W_I F_{\alpha\beta} + W_J \epsilon_{\beta\gamma}\epsilon_{\alpha\delta}F_{\delta\gamma}$

in case of material isotropy, where W_I and W_J are the partial derivatives of $W(I, J)$ with respect to its arguments. The corresponding constitutive relations for the actual stresses are obtained by means of (1.4) and, because of (1.13), may be written as

$$\underset{\sim}{\tau} = \underset{\sim}{\tau}^T = \frac{2}{J}W_I\underset{\sim}{G} + W_J\underset{\sim}{1} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \tag{1.18}$$

or $\quad \tau_{\alpha\beta} = \tau_{\beta\alpha} = \dfrac{2}{J}W_I G_{\alpha\beta} + W_J \delta_{\alpha\beta} \ , \ G_{\alpha\beta} = F_{\alpha\gamma}F_{\beta\gamma},$

if $\underset{\sim}{1}$ is the two-dimensional idem-tensor and $\delta_{\alpha\beta}$ the Kronecker-delta. According to (1.18) the tensors $\underset{\sim}{\tau}$ and $\underset{\sim}{G}$ have the same principal axes. Finally, (1.17) together with

$$\frac{\partial F_{\beta\alpha}^{-1}}{\partial F_{\gamma\delta}} = -F_{\beta\gamma}^{-1}F_{\delta\alpha}^{-1} \tag{1.19}$$

yield the appropriate specialization of (1.8) in the form

$$c_{\alpha\beta\gamma\delta} = 2W_I \delta_{\alpha\gamma}\delta_{\beta\delta} + JW_J(F_{\beta\alpha}^{-1}F_{\delta\gamma}^{-1} - F_{\delta\alpha}^{-1}F_{\beta\gamma}^{-1}) + 4W_{II}F_{\alpha\beta}F_{\gamma\delta}$$

$$+ J^2 W_{JJ}F_{\beta\alpha}^{-1}F_{\delta\gamma}^{-1} + 2JW_{IJ}(F_{\beta\alpha}^{-1}F_{\gamma\delta} + F_{\alpha\beta}F_{\delta\gamma}^{-1}) \ . \tag{1.20}$$

For a pure homogeneous plane deformation, referred to its principal

axes, (1.1) become

$$y_\alpha = \lambda_\alpha x_\alpha \text{ (no sum)}, \tag{1.21}$$

in which the coefficients λ_α are positive constants and are readily identified as the associated principal stretches. If the material is isotropic, the coordinate axes — in this particular instance — are at the same time principal axes common to the now constant symmetric tensor fields $\underset{\sim}{C}$, $\underset{\sim}{G}$, and $\underset{\sim}{\tau}$, as well as to $\underset{\sim}{\sigma}$, which at present is also symmetric. On adopting the notation

$$W_\alpha = \frac{\partial W}{\partial \lambda_\alpha} , \quad W_{\alpha\beta} = \frac{\partial^2 W}{\partial \lambda_\alpha \partial \lambda_\beta} \tag{1.22}$$

whenever W is to be regarded as a function of λ_1 and λ_2, one finds that (1.21) reduces (1.17), (1.18) to

$$\sigma_\alpha \equiv \sigma_{\alpha\alpha} = W_\alpha \text{ (no sum)}, \quad \sigma_{\alpha\beta} = 0 \ (\alpha \neq \beta), \tag{1.23}$$

$$\tau_\alpha \equiv \tau_{\alpha\alpha} = \frac{\lambda_\alpha}{\lambda_1 \lambda_2} W_\alpha \text{ (no sum)}, \quad \tau_{\alpha\beta} = 0 \ (\alpha \neq \beta), \tag{1.24}$$

where τ_α are the principal Cauchy stresses. As is easily seen, (1.24) continue to hold locally for an arbitrary plane deformation (1.1), provided λ_α and τ_α are the local principal stretches and principal Cauchy stresses. In what follows we will need to refer to the two-dimensional Baker-Ericksen inequality, according to which

$$(\tau_1 - \tau_2)(\lambda_1 - \lambda_2) > 0 \text{ if } \lambda_1 \neq \lambda_2 \tag{1.25}$$

for all pure homogeneous deformations. In view of (1.24), this condition is equivalent to

$$(\lambda_1 W_1 - \lambda_2 W_2)(\lambda_1 - \lambda_2) > 0 \text{ if } \lambda_1 \neq \lambda_2. \tag{1.26}$$

We shall from here on assume that the strain-energy density and all stresses vanish in the undeformed state and thus require that $W(\lambda_1, \lambda_2)$ obey

$$W(1, 1) = 0 \quad, \quad W_\alpha(1, 1) = 0. \tag{1.27}$$

The transition to the infinitesimal theory of elastostatic plane strain, which aims at a neighborhood of the undeformed state, is effected through a systematic linearization with respect to the displacement gradients of the finite theory recalled above. The underlying limit process, which corresponds to $\lambda_\alpha \to 1$, confirms that $\sigma_{\alpha\beta}$ and $\tau_{\alpha\beta}$ coincide to dominant order in this approximation. Upon setting

$$\overset{\bullet}{\mu} = \frac{1}{2}[W_{11}(1, 1) - W_{12}(1, 1)] \quad, \quad \overset{\bullet}{\lambda} = W_{12}(1, 1), \tag{1.28}$$

one recovers in this manner from (1.17) or (1.18) and (1.7), (1.20) the stress-displacement relations and the displacement equations of equilibrium of the linear theory for isotropic materials with $\overset{\bullet}{\mu}$ as shear modulus and $\overset{\bullet}{\lambda}$ as Lamé's modulus. Accordingly, Poisson's ratio for infinitesimal deformations is given by

$$\overset{\bullet}{\nu} = \frac{\overset{\bullet}{\lambda}}{2(\overset{\bullet}{\lambda} + \overset{\bullet}{\mu})} = \frac{W_{12}(1, 1)}{W_{11}(1, 1) + W_{12}(1, 1)}. \tag{1.29}$$

With reference to the definitions (1.10) and (1.12) we now cite from [2] necessary and sufficient conditions, in terms of restrictions upon $W(\lambda_1, \lambda_2)$ and the principal stretches λ_α, for the ordinary and strong ellipticity of the two-dimensional displacement equations of equilibrium (1.7) in the presence of material isotropy. For this purpose let

$$\Delta = \lambda_1^2 - \lambda_2^2 \ , \ \ D = \frac{1}{\Delta}(\lambda_1 W_1 - \lambda_2 W_2) \text{ if } \lambda_1 \neq \lambda_2,$$

$$D = \frac{1}{2}\left(\frac{1}{\lambda}W_1 + W_{11} - W_{12}\right) \text{ if } \lambda_1 = \lambda_2 = \lambda,$$

(1.30)

and define a symmetric matrix $\underset{\approx}{E} \equiv \underset{\approx}{E}(\lambda_1, \lambda_2)$ as follows: for all $\lambda_\alpha > 0$ such that $\lambda_1 \neq \lambda_2$,

$$E_{\alpha\alpha} = DW_{\alpha\alpha} \text{ (no sum)}$$

$$E_{12} = E_{21} = \frac{1}{2}\left\{ W_{11}W_{22} - \left[W_{12} + \frac{W_1 - W_2}{\lambda_1 - \lambda_2} \right]\left[W_{12} - \frac{W_1 + W_2}{\lambda_1 + \lambda_2} \right]\right\};$$

(1 31)

for all $\lambda_\alpha > 0$ such that $\lambda_1 = \lambda_2 = \lambda$,

$$E_{\alpha\beta} = DW_{11}.$$

(1.32)

Bearing in mind the symmetry of $W(\lambda_1, \lambda_2)$ and taking for granted that this function is twice continuously differentiable, one sees that $D(\lambda_1, \lambda_2)$ and $\underset{\approx}{E}(\lambda_1, \lambda_2)$ defined by (1.31), (1.32) are continuous on the entire first quadrant of the principal-stretch plane.

As shown in Section 2 of [2], for an isotropic material the system (1.7) is <u>elliptic</u> at a solution and at a particular material point if and only if the corresponding local principal stretches satisfy the inequalities

$$E_{11}E_{22} > 0 \ , \ \ \eta E_{12} + \sqrt{E_{11}E_{22}} > 0,$$

(1.33)

where

$$\eta = \text{sgn} E_{\alpha\alpha} \text{ (no sum)};$$

(1.34)

further, necessary and sufficient for the <u>strong ellipticity</u> of this system is that

$$D > 0 , \quad E_{11} > 0 , \quad E_{22} > 0 , \quad E_{12} + \sqrt{E_{11} E_{22}} > 0. \tag{1.35}$$

Note from (1.30) that $D > 0$, if $\lambda_1 \neq \lambda_2$, is equivalent to the Baker-Ericksen inequality (1.26), the latter being a necessary condition for strong ellipticity. Moreover, (1.35) evidently imply (1.33), as should be the case.

At the undeformed state, characterized by $\lambda_1 = \lambda_2 = 1$, the strong-ellipticity conditions (1.35) by virtue of (1.32), (1.30), and (1.27) reduce to

$$W_{11}(1,1) > 0 , \quad W_{11}(1,1) - W_{12}(1,1) > 0. \tag{1.36}$$

These inequalities, in turn, because of (1.28), become

$$\dot{\mu} > 0 , \quad \dot{\lambda} + 2\dot{\mu} > 0, \tag{1.37}$$

which are precisely the familiar conditions for the strong ellipticity of the linearized displacement equations of equilibrium in case of isotropy. We proved at the end of Section 2 in [2] that when (1.37) hold true — so that strong ellipticity prevails at infinitesimal deformations — every open connected set of ordinary ellipticity in the (λ_1, λ_2)-plane that contains the undeformed state $(1,1)$ is necessarily also a domain of strong ellipticity.

Finally, suppose ellipticity has failed, so that (1.33) are violated at some material point, for a particular plane deformation. Then there must exist at least one (real) characteristic curve passing through this point. We call such a curve a "material characteristic" when it is referred to the undeformed configuration and apply the term "spatial characteristic" to the deformation image of a material characteristic. Let γ be the angle of inclination of a local spatial characteristic relative to the first principal axis of the local Cauchy stress tensor $\underset{\sim}{\tau}$ and hence also of the left deformation

tensor $\underset{\sim}{G}$. As shown in Section 3 of [2], $\cos 2\gamma$ is a solution of

$$(\lambda_1^4 E_{11} + \lambda_2^4 E_{22} - 2\lambda_1^2\lambda_2^2 E_{12})\cos^2 2\gamma$$

$$+2(\lambda_2^4 E_{22} - \lambda_1^4 E_{11})\cos 2\gamma + (\lambda_1^4 E_{11} + \lambda_2^4 E_{22} + 2\lambda_1^2\lambda_2^2 E_{12}) = 0. \qquad (1.38)$$

where $\underset{\sim}{E}(\lambda_1,\lambda_2)$ is the symmetric matrix introduced in (1.31), (1.32). The discriminant of this quadratic equation is non-negative throughout the complement $\mathcal{N}$ of the domain of ellipticity $\mathcal{E}$, characterized by (1.33), with respect to the open first quadrant of the (λ_1,λ_2)-plane.[1] In the _interior_ of $\mathcal{N}$ equation (1.38) in general has two distinct real roots within the interval $[-1,1]$, corresponding to four distinct spatial characteristic directions. The latter evidently occur in two pairs, each symmetrically situated with respect to $\gamma = 0$. In contrast, for points on the common boundary of $\mathcal{E}$ and $\mathcal{N}$ (ellipticity boundary), which are associated with an "incipient failure of ellipticity", there is at most one such pair of characteristic directions, the two real roots of (1.38) being necessarily coalescent.[2]

2. Two special elastic materials and some of their response properties.

For future illustrative purposes we turn here to a class of homogeneous and isotropic elastic materials, whose in-plane response to a plane deformation is governed by an elastic potential of the form

$$W(I, J) = \frac{\mu}{2}[If(J) + g(J)] \quad (\mu > 0), \qquad (2.1)$$

---

[1] The set $\mathcal{N}$ of non-elliptic points in the principal-stretch plane may of course be empty for a particular elastic material.

[2] See the exhaustive discussion of incipient failures of ellipticity at the end of Section 3 in [2]. In this degenerate case there may exist only a single characteristic.

in which μ is a material constant. Note that this potential depends linearly upon the invariant I. Since the corresponding $W(\lambda_1, \lambda_2)$ is to satisfy (1.27), the functions f and g are subject to the requirements

$$2f(1)+g(1)=0 \ , \ 2f(1)+2f'(1)+g'(1)=0, \tag{2.2}$$

the primes denoting differentiation. For every fixed choice of the functions f and g consistent with (2.2) one can readily exhibit an infinite class of three-dimensional elastic potentials, each of which yields (2.1) upon specialization to plane strain.

We now particularize (2.1) in two different ways and hereafter refer to the corresponding materials as Material 1 and Material 2. Let

$$\left. \begin{array}{l} f(J)=J^{-2} \ , \ g(J)=2J-4 \text{ for Material 1}, \\[2ex] f(J)=J \ , \ g(J)=8J^{-\frac{1}{2}}-10 \text{ for Material 2}, \end{array} \right\} \tag{2.3}$$

Accordingly, for Material 1,

$$W(\lambda_1, \lambda_2) = \frac{\mu}{2}[2\lambda_1\lambda_2 + \lambda_1^{-2} + \lambda_2^{-2} - 4], \tag{2.4}$$

whereas for Material 2,

$$W(\lambda_1, \lambda_2) = \frac{\mu}{2}[\lambda_1^3\lambda_2 + \lambda_1\lambda_2^3 + 8(\lambda_1\lambda_2)^{-\frac{1}{2}} - 10] . \tag{2.5}$$

Both materials conform to (2.2), whence the energy density and all stresses vanish in the undeformed state. Further, the first of (1.28) gives $\mu = \overset{\bullet}{\mu}$ in either instance, so that μ is the shear modulus for infinitesimal deformations; on the other hand, the second of (1.28), together with (1.29), furnish $\overset{\bullet}{\lambda} = 2\mu, \overset{\bullet}{\nu} = \frac{1}{4}$ for Material 1 and $\overset{\bullet}{\lambda} = 4\mu, \overset{\bullet}{\nu} = \frac{2}{5}$ for Material 2. Therefore, (1.37) assure strong ellipticity at infinitesimal deformations for both materials.

In contrast, as will become apparent in what follows, neither material remains elliptic at all plane deformations.

Our next objective is a detailed discussion of relevant response properties pertaining to Material 1. In this connection we note first that $W(\lambda_1, \lambda_2)$ in (2.4) is the restriction to plane strain of an elastic potential proposed by Blatz and Ko [3] in an attempt to match experimental data obtained in tests of a highly compressible rubberlike material. The idealized material thus adopted in [3] was the subject of [1], which contains a comprehensive three-dimensional treatment of its response characteristics and ellipticity restrictions. Since [1] also deals explicitly with the special case of plane deformations to which we confine our attention at present, the subsequent results are mostly cited from [1] without intermediate detail.

For a pure homogeneous plane deformation of the form (1.21) one infers from (2.4) and (1.23), (1.24) the in-plane response

$$\tau_\alpha = \mu[1 - (\lambda_1\lambda_2)^{-1}\lambda_\alpha^{-2}] \ , \ \sigma_\alpha = \mu(\lambda_1\lambda_2\lambda_\alpha^{-1} - \lambda_\alpha^{-3}) \tag{2.6}$$

appropriate to Material 1. Specializing (2.6) for the case of isotropic plane strain, one has

$$\lambda_1 = \lambda_2 \equiv \lambda \ , \ \tau_1 = \tau_2 \equiv \tau = \mu(1 - \lambda^{-4}) \ , \ \sigma_1 = \sigma_2 \equiv \sigma = \mu(\lambda - \lambda^{-3}). \tag{2.7}$$

In the particular instance of plane-strain uni-axial stress, parallel to the x_1-axis, (2.6) furnish

$$\tau_2 = \sigma_2 = 0 \ , \ \lambda_2 = \lambda_1^{-\frac{1}{3}} \ , \ \tau_1 = \mu(1 - \lambda_1^{-\frac{8}{3}}) \ , \ \sigma_1 = \mu(\lambda_1^{-\frac{1}{3}} - \lambda_1^{-3}). \tag{2.8}$$

We consider now a homogeneous plane deformation corresponding to a state of simple shear of the form

$$y_1 = x_1 + \varkappa x_2 \ , \ y_2 = x_2, \qquad (2.9)$$

in which $\varkappa$ is a constant, $\tan^{-1}\varkappa$ being the angle of shear. The principal
stretches of this deformation are related to $\varkappa$ through

$$\lambda_1 = \sqrt{1 + \frac{\varkappa^2}{2} + \varkappa\sqrt{1 + \frac{\varkappa^2}{4}}} \ , \ \lambda_2 = \frac{1}{\lambda_1}. \qquad (2.10)$$

The response of Material 1 to such a simple shear is readily deduced with
the aid of (1.17) and (1.18). In this manner one arrives at

$$\left. \begin{array}{l} \tau_{11} = 0 \ , \ \tau_{22} = -\mu\varkappa^2 \ , \ \tau_{12} = \tau_{21} = \mu\varkappa, \\[2ex] \sigma_{11} = \sigma_{22} = -\mu\varkappa^2 \ , \ \sigma_{12} = \mu\varkappa \ , \ \sigma_{21} = \mu(\varkappa + \varkappa^3). \end{array} \right\} \qquad (2.11)$$

As is clear from (2.7), in <u>isotropic plane strain</u>, $\tau(\lambda)$ and $\sigma(\lambda)$ are
concave, monotone increasing functions for $0 < \lambda < \infty$ and $\tau(\lambda) \to -\infty, \sigma(\lambda) \to -\infty$ as
$\lambda \to 0$, but $\tau(\lambda) \to \mu, \sigma(\lambda) \to \infty$ as $\lambda \to \infty$. The response of Material 1 to <u>plane-strain</u>
<u>uni-axial stress</u> is depicted in Figure 1. According to (2.8), here $\tau_1(\lambda_1)$ is a
concave, monotone increasing function for $0 < \lambda_1 < \infty$ and $\tau_1(\lambda_1) \to -\infty$ as $\lambda_1 \to 0$,
while $\tau_1(\lambda_1) \to \mu$ as $\lambda_1 \to \infty$. In contrast, the axial nominal stress $\sigma_1(\lambda_1)$ in-
creases steadily from $\sigma_1(0+) = -\infty$ to a positive maximum at $\lambda_1 = 3^{\frac{3}{4}} \doteq 2.28$ and
thereafter diminishes steadily toward zero. The behavior of Material 1 in
<u>simple shear</u> is immediate from (2.11). Both the actual shear stress τ_{12}
and the nominal shear stress σ_{12} are directly proportional to the amount of
shear $\varkappa$, but $\sigma_{21}(\varkappa)$ is a convex strictly increasing function for $0 < \varkappa < \infty$. A
significant nonlinear effect is reflected in the fact that the shear deforma-
tion (2.9) induces an actual normal stress τ_{22}; the latter is <u>compressive</u>

for the material under consideration.

From (2.4), (1.30), (1.31), as well as the continuity of $D(\lambda_1, \lambda_2)$ and $E_{\alpha\beta}(\lambda_1, \lambda_2)$, follows for Material 1 at all $\lambda_\alpha > 0$,

$$D = \frac{\mu}{\lambda_1^2 \lambda_2^2} > 0 , \quad E_{11} = \frac{3\mu^2}{\lambda_1^6 \lambda_2^2} > 0 , \quad E_{22} = \frac{3\mu^2}{\lambda_1^2 \lambda_2^6} > 0 ,$$

$$E_{12} = \frac{\mu^2}{2\lambda_1^6 \lambda_2^6} (8\lambda_1^2 \lambda_2^2 - \lambda_1^4 - \lambda_2^4) . \tag{2.12}$$

Hence $\eta = 1$ in (1.34) and the first of the conditions of ordinary ellipticity (1.33) holds true for all plane deformations of this material, as does the Baker-Ericksen inequality (1.26). On the other hand, (2.12), the second of (1.33), and (1.14) reveal that ellipticity prevails at present if and only if

$$14\lambda_1^2 \lambda_2^2 - \lambda_1^4 - \lambda_2^4 > 0 \quad \text{or} \quad \frac{I}{J} < 4, \tag{2.13}$$

either of which is equivalent to

$$\rho < \frac{\lambda_1}{\lambda_2} < \frac{1}{\rho} , \quad \rho = 2 - \sqrt{3}. \tag{2.14}$$

Moreover, in view of (1.35), these conditions are also necessary <u>and</u> <u>suffi-cient</u> for strong ellipticity.

Let (r, θ) be polar coordinates in the (λ_1, λ_2)-plane, defined by

$$\lambda_1 = r\cos\theta , \quad \lambda_2 = r\sin\theta \quad (0 < r < \infty, 0 < \theta < \pi/2). \tag{2.15}$$

Then (2.14) may be written as

$$0 < r < \infty , \quad \frac{\pi}{12} < \theta < \frac{5\pi}{12}. \tag{2.16}$$

This wedge-shaped domain of ellipticity $\mathcal{E}$ in the principal-stretch plane is

shown in Figure 2. The figure also displays the "isotropic extension path" $\lambda_1 = \lambda_2$, the "uni-axial stress path" $\lambda_2 = \lambda_1^{-\frac{1}{3}}$, and the "simple-shear path" $\lambda_2 = \lambda_1^{-1}$, supplied by (2.7), (2.8), and (2.10). Since the first of these paths lies wholly in $\mathcal{E}$, Material 1 does not suffer a loss of ellipticity in <u>isotropic extension</u>. On the other hand, each of the other two paths intersects the boundary of $\mathcal{E}$ twice. Ellipticity is seen to fail in both <u>uni-axial tension</u> <u>and</u> <u>compression</u> according as

$$\lambda_1 \geq \rho^{-\frac{3}{4}} \doteq 2.68 \text{ or } \lambda_1 \leq \rho^{\frac{3}{4}} \doteq 0.37; \tag{2.17}$$

the limits of ellipticity for uni-axial stress are also marked in Figure 1. Finally, in <u>simple shear</u> a breakdown in ellipticity is found to occur whenever either principal stretch equals or exceeds the value $\rho^{-\frac{1}{2}} \doteq 1.93$, which corresponds to a shear angle of approximately $55°$.

According to (1.38) and (2.12), <u>the inclination of the spatial characteristics</u> relative to the first principal axis of Cauchy stress, for Material 1 obeys

$$\cos 2\gamma = \pm \frac{\sqrt{\lambda_1^4 + \lambda_2^4 - 14\lambda_1^2 \lambda_2^2}}{\lambda_1^2 - \lambda_2^2}. \tag{2.18}$$

When ellipticity has failed, so that (2.13) is violated, (2.18) evidently furnishes two distinct pairs of real characteristic directions, except on the boundary of $\mathcal{E}$, where $\gamma = \pm \pi/4$ are the only solutions of (2.18) within $-\pi < \gamma \leq \pi$. Thus, at an incipient failure of ellipticity the spatial characteristic directions for Material 1 coincide with the local directions of the lines of maximum and minimum actual shearing stress. Because of (2.15), equation (2.18) alternatively becomes

$$\cos 2\gamma = \pm \sqrt{\frac{1 - 4\sin^2 2\theta}{\cos 2\theta}} \ , \qquad\qquad (2.19)$$

from which one infers that the characteristic directions do not vary along any ray θ=constant in the non-elliptic part of the (λ_1, λ_2)-plane. This completes the discussion of Material 1.

A parallel discussion of <u>Material</u> 2, the plane-strain elastic potential of which is given by (2.5), is analytically more awkward. Here (2.6) give way to

$$\left.\begin{array}{l} \tau_\alpha = \dfrac{\mu}{2}\left[\, 3\lambda_\alpha^2 + (\lambda_1\lambda_2)^2 \lambda_\alpha^{-2} - 4(\lambda_1\lambda_2)^{-\frac{3}{2}} \,\right], \\[4mm] \sigma_\alpha = \dfrac{\mu}{2}\left[\, 3\lambda_1\lambda_2\lambda_\alpha + (\lambda_1\lambda_2)^3 \lambda_\alpha^{-3} - 4(\lambda_1\lambda_2)^{-\frac{1}{2}}\lambda_\alpha^{-1} \,\right]. \end{array}\right\} \qquad (2.20)$$

Hence for <u>isotropic plane strain</u>,

$$\lambda_1 = \lambda_2 \equiv \lambda \ , \ \ \tau_1 = \tau_2 \equiv \tau = 2\mu(\lambda^2 - \lambda^{-3}) \ , \ \ \sigma_1 = \sigma_2 \equiv \sigma = 2\mu(\lambda^3 - \lambda^{-2}). \qquad (2.21)$$

In the case of <u>plane-strain uni-axial stress</u>, parallel to the x_1-axis, one has by virtue of the first of (2.20),

$$\tau_2 = \sigma_2 = 0 \ , \ \ 3\lambda_2^2 + \lambda_1^2 - 4(\lambda_1\lambda_2)^{-\frac{3}{2}} = 0, \qquad\qquad (2.22)$$

the last of which implicitly determines the transverse stretch λ_2 as a function of the axial stretch λ_1. Thus,

$$\lambda_2 = \overline{\lambda}(\lambda_1) \ \ (0 < \lambda_1 < \infty) \qquad\qquad (2.23)$$

and (2.20), (2.22), (2.23) lead to

$$\tau_1 = \mu[\lambda_1^2 - \overline{\lambda}^2(\lambda_1)] \ , \ \ \sigma_1 = \mu[\lambda_1^2\overline{\lambda}(\lambda_1) - \overline{\lambda}^3(\lambda_1)]. \qquad (2.24)$$

Although $\overline{\lambda}(\lambda_1)$ is not obtainable in elementary form, one shows without difficulty that this function is strictly decreasing. Further, it is easy to deduce an explicit parametric representation, in terms of the polar angle θ introduced in (2.15), for the curve corresponding to (2.23). This representation was used to plot the uni-axial tension path appearing in Figure 4, as well as the stress-stretch curves depicted in Figure 3. For a simple shear deformation, characterized by (2.9), (2.10), one finds at present,

$$\tau_{11} = \frac{3\mu \varkappa^2}{2} \ , \ \ \tau_{22} = \frac{\mu \varkappa^2}{2} \ , \ \ \tau_{12} = \tau_{21} = \mu \varkappa,$$

$$\sigma_{11} = \sigma_{22} = \frac{\mu \varkappa^2}{2} \ , \ \ \sigma_{12} = \mu \varkappa \ , \ \ \sigma_{21} = \mu \left(\varkappa - \frac{\varkappa^3}{2} \right). \tag{2.25}$$

The preceding equations reveal certain essential qualitative differences between Material 2 and Material 1, as far as their response to the special homogeneous deformations under consideration is concerned. The functions $\tau(\lambda)$ and $\sigma(\lambda)$ in (2.21), which govern the response of Material 2 to isotropic plane strain, are both monotone increasing and, as is the case for Material 1, $\tau(\lambda) \to -\infty, \sigma(\lambda) \to -\infty$ for $\lambda \to 0$; however, now both the actual and the nominal stress tend to infinity as $\lambda \to \infty$. Similarly, it is clear from Figure 3, which pertains to uni-axial stress, that the true axial stress $\tau_1(\lambda_1) \to \infty$ as $\lambda_1 \to \infty$, while the behavior of the nominal stress $\sigma_1(\lambda_1)$ is qualitatively the same as in Figure 1. Further, we observe on the basis of (2.25) that a simple shear of Material 2 induces non-zero actual stresses τ_{11} and τ_{22}, both of which are tensile.

In place of (2.12) one obtains for Material 2 at all $\lambda_\alpha > 0$,

$$D = \mu \lambda_1 \lambda_2 > 0,$$

$$E_{\alpha\alpha} = 3\mu^2 \left[\lambda_1^2 \lambda_2^2 + (\lambda_1 \lambda_2)^{\frac{1}{2}} \lambda_\alpha^{-2} \right] > 0 \text{ (no sum)},$$

$$E_{12} = \frac{\mu^2}{2} \left[10\lambda_1^2 \lambda_2^2 - \lambda_1^2 - \lambda_2^2 + 3(\lambda_1^2 + \lambda_2^2)(\lambda_1 \lambda_2)^{-\frac{3}{2}} \right].$$

$$(2.26)$$

In view of (2.26), (1.30), the Baker-Ericksen inequality (1.26) is satisfied also for Material 2. Moreover, an appeal to (1.33), (1.34), (1.35) confirms that ordinary and strong ellipticity once again prevail if and only if the last of (1.35) holds true. The latter may, with the aid of (2.26) and (2.15), be written as

$$(10\omega^2 - 1)\zeta + 3 + 6\omega \sqrt{\omega^2 \zeta^2 + \zeta + 1} > 0, \qquad (2.27)$$

provided one sets

$$\omega = \frac{1}{2}\sin 2\theta, \quad \zeta = r^5 \omega^{\frac{3}{2}}. \qquad (2.28)$$

Upon exclusion of the extraneous root of the quadratic equation in ζ obtained by squaring (2.27), one arrives at a necessary and sufficient condition of ellipticity for Material 2 in the form

$$r < \left[\frac{6\sqrt{2}}{(1 - 2\sin 2\theta)(\sin 2\theta)^{\frac{3}{2}}} \right]^{\frac{1}{6}} \quad \left(0 < \theta < \frac{\pi}{12}, \ \frac{5\pi}{12} < \theta < \frac{\pi}{2} \right), \qquad (2.29)$$

where (r, θ) are the polar coordinates in the principal-stretch plane, defined by (2.15).

Figure 4, which is the counterpart for Material 2 of Figure 2, shows the domain of ellipticity $\mathcal{E}$ appropriate to the second material. This figure also exhibits the deformation paths in the (λ_1, λ_2)-plane corresponding to isotropic extension, simple shear, and uni-axial stress — the first two of

which are, of course, the same as in Figure 2. Here again ellipticity never fails in isotropic extension and breaks down for all sufficiently severe simple shears. In contrast to the results for Material 1, however, those given in Figure 4 reveal the analytically verifiable fact that Material 2 cannot suffer a loss of ellipticity in uni-axial <u>compression</u>; the failure of ellipticity in uni-axial <u>tension</u> is seen to occur at an axial stretch $\lambda_1 \doteq 3.04$ (see also Figure 3).

The inclination γ of the real <u>spatial</u> <u>characteristics</u> for Material 2, whose existence is assured if the ellipticity condition (2.29) is violated, follows from (1.38) by recourse to (2.26) and (2.15). In this manner one is led to

$$\cos 2\gamma = \frac{1}{4\zeta\cos 2\theta}\left[3(1+\zeta)\pm\sqrt{(\zeta-3)^2 - 16w^2\zeta^2}\right], \tag{2.30}$$

w and ζ being the auxiliary functions of the polar coordinates (r, θ) adopted in (2.28). Equation (2.30) is the analogue for Material 2 of (2.19). At all points in the complement of $\mathcal{E}$ with respect to the open first quadrant of the (λ_1, λ_2)-plane, except for the points on the boundary of $\mathcal{E}$, (2.30) yield four distinct characteristic directions. On the boundary of $\mathcal{E}$ there results a single pair of distinct characteristics, whose inclincations are determined by

$$\cos 2\gamma = \frac{2 - \sin 2\theta}{2\cos 2\theta}, \tag{2.31}$$

where θ is the angular polar coordinate of the boundary point in question. Thus, the characteristic directions at an incipient failure of ellipticity of Material 2, unlike those associated with Material 1, are no longer constant. In particular, at an incipient breakdown of ellipticity in uni-axial tension parallel to the x_1-axis, one has $\theta \doteq 3.51°$ (see Figure 4) and (2.31) yields $\gamma \doteq \pm 9.46°$.

3. **Elastostatic fields with discontinuous deformation gradients. Piecewise homogeneous elastostatic shocks.**

At this stage we turn to our main objective, which concerns the existence and nature of a class of elastostatic fields with discontinuous deformation gradients appropriate to a homogeneous, but not necessarily isotropic, hyperelastic solid. Suppose such a solid, in its undeformed configuration, occupies the entire (x_1, x_2)-plane $\mathcal{R}$ spanned by a rectangular cartesian coordinate frame X. Let $\mathcal{L}$, with the unit direction vector $\underset{\sim}{L}$, be a straight line through the origin of X, so that

$$\mathcal{L}: \underset{\sim}{x} = \hat{\underset{\sim}{x}}(\xi) = \underset{\sim}{L}\xi \quad (-\infty < \xi < \infty), \tag{3.1}$$

and call $\underset{\sim}{N}$ the unit normal vector of $\mathcal{L}$ obtained by a counter-clockwise rotation of $\underset{\sim}{L}$ through a right angle. Next, we designate by $\overset{+}{\mathcal{R}}$ and $\overset{-}{\mathcal{R}}$ the two open half-planes into which $\mathcal{L}$ divides $\mathcal{R}$, with the understanding that $\underset{\sim}{N}$ points into $\overset{+}{\mathcal{R}}$ (Figure 5). Consider now a piecewise homogeneous plane deformation of the form

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x}) = \begin{cases} \overset{+}{\underset{\sim}{F}}\underset{\sim}{x} \text{ for all } \underset{\sim}{x} \text{ in } \overset{+}{\mathcal{R}}, & \overset{+}{J} = \det\overset{+}{\underset{\sim}{F}} > 0 \\[2ex] \overline{\underset{\sim}{F}}\underset{\sim}{x} \text{ for all } \underset{\sim}{x} \text{ in } \overline{\mathcal{R}}, & \overline{J} = \det\overline{\underset{\sim}{F}} > 0. \end{cases} \tag{3.2}$$

Here $\overset{+}{\underset{\sim}{F}}$ and $\overline{\underset{\sim}{F}}$ are constant (nonsingular) tensors, which evidently represent the position-independent deformation-gradient fields prevailing on $\overset{+}{\mathcal{R}}$ and $\overline{\mathcal{R}}$, respectively, while $\overset{+}{J}$ and $\overline{J}$ are the corresponding Jacobian determinants.

According to (1.6), the nominal stress field induced by the deformation (3.2) is given by

$$\overset{+}{\underset{\sim}{\sigma}} = W_{\underset{\sim}{F}}(\overset{+}{\underset{\sim}{F}}) \text{ on } \overset{+}{\mathcal{R}}, \quad \overline{\underset{\sim}{\sigma}} = W_{\underset{\sim}{F}}(\overline{\underset{\sim}{F}}) \text{ on } \overline{\mathcal{R}} \tag{3.3}$$

and automatically satisfies the equilibrium equation (1.5) on either side of $\mathcal{L}$. We shall assume that the displacement field associated with (3.2) is continuous across $\mathcal{L}$, so that

$$\overset{+}{\mathbf{F}}\underset{\sim}{\mathbf{x}}=\overset{-}{\mathbf{F}}\underset{\sim}{\mathbf{x}} \text{ for every } \underset{\sim}{\mathbf{x}} \text{ on } \mathcal{L}, \tag{3.4}$$

which — in view of (3.1) — is equivalent to

$$\overset{+}{\underset{\sim}{\mathbf{F}}}\underset{\sim}{\mathbf{L}}=\overset{-}{\underset{\sim}{\mathbf{F}}}\underset{\sim}{\mathbf{L}}. \tag{3.5}$$

This assumption permits us to extend the mapping (3.2) continuously onto $\mathcal{L}$ and excludes from our present considerations any separation or gliding of material along the unique deformation image $\mathcal{L}_*$ of $\mathcal{L}$. Moreover, as is clear from (3.1), (3.5),

$$\mathcal{L}_*: \underset{\sim}{\mathbf{y}}=\hat{\underset{\sim}{\mathbf{y}}}(\xi)=\overset{+}{\underset{\sim}{\mathbf{F}}}\underset{\sim}{\mathbf{L}}\xi=\overset{-}{\underset{\sim}{\mathbf{F}}}\underset{\sim}{\mathbf{L}}\xi \quad (-\infty<\xi<\infty). \tag{3.6}$$

Figure 5(b) illustrates typical deformation images of the three rectangles shown in Figure 5(a) under the mapping (3.2), subject to the continuity requirement (3.4).

Finally, equilibrium, i.e. the balance of forces across $\mathcal{L}$, demands the continuity of the Piola tractions at $\mathcal{L}$, so that

$$\overset{+}{\underset{\sim}{\sigma}}\underset{\sim}{\mathbf{N}}=\overset{-}{\underset{\sim}{\sigma}}\underset{\sim}{\mathbf{N}}\equiv\underset{\sim}{\mathbf{s}}, \tag{3.7}$$

where $\underset{\sim}{\mathbf{s}}$ is the nominal traction exerted on the material in $\overset{-}{\mathcal{R}}$ by the material occupying $\overset{+}{\mathcal{R}}$ in the undeformed configuration.

When $\overset{+}{\underset{\sim}{\mathbf{F}}}\neq\overset{-}{\underset{\sim}{\mathbf{F}}}$, we shall refer to the elastostatic field characterized by (3.2), (3.3), together with the continuity conditions (3.5), (3.7), as a piece-wise homogeneous elastostatic shock (equilibrium shock); further, we shall

call the straight lines $\mathcal{L}$ and $\mathcal{L}_*$, governed by (3.1) and (3.6), the <u>material</u> <u>and</u> <u>the</u> <u>spatial</u> <u>shock-line</u>.

The first question that arises in connection with piecewise homogeneous elastostatic shocks as characterized above concerns their existence: are there such shocks for a given hyperelastic material and a given deformation gradient $\overset{+}{\underset{\sim}{F}}$? If so, how many? Is their existence contingent upon restrictions on the governing elastic potential? Further, in case such shocks exist, what are the corresponding orientations of the shock-lines and values of $\overset{-}{\underset{\sim}{F}}$? Also, what is the nature of the emerging elastostatic fields and what kind of field discontinuities at the shock-lines do they entail?

Before we can attempt to find at least partial answers to the foregoing questions we need to explore in some detail the kinematics of equilibrium shocks. To this end we designate by $\underset{\sim}{\ell}$ the unit direction vector of $\mathcal{L}_*$, note on the basis of (3.6) that

$$\underset{\sim}{\ell} = c\overset{+}{\underset{\sim}{F}}\underset{\sim}{L} = c\overset{-}{\underset{\sim}{F}}\underset{\sim}{L} \ , \quad c = \frac{1}{|\overset{+}{\underset{\sim}{F}}\underset{\sim}{L}|} = \frac{1}{|\overset{-}{\underset{\sim}{F}}\underset{\sim}{L}|} \ , \tag{3.8}$$

and assign to $\mathcal{L}_*$ the unit normal vector $\underset{\sim}{n}$ resulting from a counter-clockwise rotation of $\underset{\sim}{\ell}$ through $\pi/2$. Clearly, $\mathcal{L}_*$ separates the two open half-planes $\overset{+}{\underset{\sim}{R}}_*$ and $\overset{-}{\underset{\sim}{R}}_*$ that are the deformation images of $\overset{+}{\underset{\sim}{R}}$ and $\overset{-}{\underset{\sim}{R}}$; also, $\underset{\sim}{n}$ points into $\overset{+}{\underset{\sim}{R}}_*$(see Figure 5). Now let Φ and φ stand for the angles of inclination, relative to the x_1-axis, of $\mathcal{L}$ and $\mathcal{L}_*$, respectively. Both of these angles, hereafter referred to as the <u>material</u> <u>and</u> <u>spatial</u> <u>shock-angles</u>, may be confined to the interval $[-\pi/2, \pi/2]$ and

$$\left.\begin{array}{l} L_1 = \cos\Phi \ , \ L_2 = \sin\Phi \ , \ N_1 = -\sin\Phi \ , \ N_2 = \cos\Phi, \\[2mm] \ell_1 = \cos\varphi \ , \ \ell_2 = \sin\varphi \ , \ n_1 = -\sin\varphi \ , \ n_2 = \cos\varphi, \end{array}\right\} \tag{3.9}$$

$$L_\alpha = \epsilon_{\alpha\beta} N_\beta \; , \; N_\alpha = \epsilon_{\beta\alpha} L_\beta \; , \; \ell_\alpha = \epsilon_{\alpha\beta} n_\beta \; , \; n_\alpha = \epsilon_{\beta\alpha} \ell_\beta . \tag{3.10}$$

Further, from (3.8), (3.10) and the last of (1.16) follows

$$\underset{\sim}{L} = \frac{1}{c} (\overset{+}{\underset{\sim}{F}})^{-1} \underset{\sim}{\ell} = \frac{1}{c} (\bar{\underset{\sim}{F}})^{-1} \underset{\sim}{\ell} \; , \; \underset{\sim}{N} = \frac{1}{c\overset{+}{J}} \overset{+}{\underset{\sim}{F}}{}^T \underset{\sim}{n} = \frac{1}{c\bar{J}} \bar{\underset{\sim}{F}}{}^T \underset{\sim}{n} . \tag{3.11}$$

Because of (1.13), the left deformation tensors of the homogeneous deformations on $\overset{+}{\mathscr{R}}$ and $\bar{\mathscr{R}}$ are given by

$$\overset{+}{\underset{\sim}{G}} = \overset{+}{\underset{\sim}{F}}\,\overset{+}{\underset{\sim}{F}}{}^T \; , \; \bar{\underset{\sim}{G}} = \bar{\underset{\sim}{F}}\,\bar{\underset{\sim}{F}}{}^T , \tag{3.12}$$

whence, squaring the last of (3.11), one has

$$c^2 = \frac{\underset{\sim}{n} \cdot \underset{\sim}{G} \underset{\sim}{n}}{\overset{+}{J}{}^2} = \frac{\underset{\sim}{n} \cdot \underset{\sim}{G} \underset{\sim}{n}}{\bar{J}{}^2} . \tag{3.13}$$

Also, (3.8) and (3.9) yield the relation

$$\tan\varphi = \frac{\overset{+}{F}_{21} + \overset{+}{F}_{22} \tan\Phi}{\overset{+}{F}_{11} + \overset{+}{F}_{12} \tan\Phi} \tag{3.14}$$

between the material and the spatial shock-angle.

Since $\overset{+}{\underset{\sim}{F}}$ and $\bar{\underset{\sim}{F}}$ are, by hypothesis, nonsingular tensors with positive determinants, there is a tensor $\underset{\sim}{B}$ such that

$$\underset{\sim}{B} = \bar{\underset{\sim}{F}} (\overset{+}{\underset{\sim}{F}})^{-1} \; , \; \bar{\underset{\sim}{F}} = \underset{\sim}{B}\overset{+}{\underset{\sim}{F}} \; , \; \det\underset{\sim}{B} > 0. \tag{3.15}$$

One may therefore resolve the mapping (3.2) into the two successive plane deformations:

$$\underset{\sim}{z} = \overset{+}{\underset{\sim}{F}} \underset{\sim}{x} \text{ for all } \underset{\sim}{x} \text{ in } \mathscr{R}, \tag{3.16}$$

$$\chi = \begin{cases} \underset{\sim}{z} \text{ for all } \underset{\sim}{z} \text{ in } \overset{+}{\aleph}_* \\[2ex] \underset{\sim}{B} \underset{\sim}{z} \text{ for all } \underset{\sim}{z} \text{ in } \bar{\aleph}_*. \end{cases} \tag{3.17}$$

We shall call the mapping (3.16), which is homogeneous on the entire plane $\aleph$, the _intermediate deformation_ belonging to the original deformation (3.2) and refer to (3.17) as the _supplementary deformation_. The displacement continuity condition (3.4) evidently demands that the supplementary deformation carry $\mathcal{L}_*$ into itself, i.e.,

$$\underset{\sim}{B} \underset{\sim}{\ell} = \underset{\sim}{\ell}, \tag{3.18}$$

as is also apparent from (3.5) and (3.15), (3.11). At this stage we introduce a second coordinate frame X' with the same origin and the unit base vectors $(\underset{\sim}{\ell}, \underset{\sim}{n})$, so that X' is obtained by a rotation of the original frame X through the spatial shock-angle φ (see Figure 5). On referring (3.18) to the frame X' one finds at once that the displacement field of the deformation (3.2) is continuous if and only if

$$\underset{\sim}{B}' \equiv (B'_{\alpha\beta}) = \begin{bmatrix} 1 & \varkappa \\ 0 & \delta \end{bmatrix}, \tag{3.19}[1]$$

where $\varkappa$ and δ are two as yet arbitrary constants, whose kinematic significance will emerge presently. Meanwhile we take note of the matrix relation

$$\underset{\sim}{B} = \underset{\sim}{R}^T \underset{\sim}{B}' \underset{\sim}{R}, \quad \overset{+}{\underset{\sim}{G}}{}' = \underset{\sim}{R} \overset{+}{\underset{\sim}{G}} \underset{\sim}{R}^T, \quad \underset{\sim}{R} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \tag{3.20}$$

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[1] Boldface letters carrying a prime stand exclusively for the _matrix_ of scalar components in X' of the corresponding tensor or vector.

and thus conclude with the aid of (3.9) that (3.19) is equivalent to

$$B_{\alpha\beta} = \ell_\alpha \ell_\beta + \varkappa \ell_\alpha n_\beta + \delta n_\alpha n_\beta. \tag{3.21}$$

Furthermore, from (3.2), (3.15), (3.20) follows

$$\bar{J} = \det \bar{F} = \det B \det \overset{+}{F} = J \det B', \tag{3.22}$$

so that by virtue of (3.19),

$$\bar{J} = \delta \overset{+}{J}, \quad \overset{+}{J} = \lambda_1 \lambda_2, \tag{3.23}[1]$$

if λ_1, λ_2 are the principal stretches of the deformation on $\overset{+}{R}$ and hence also of the intermediate deformation (3.16). Next, (3.12), (3.15) give

$$\bar{G} = B \overset{+}{G} B^T, \quad \bar{G}' = B' \overset{+}{G}' (B')^T, \tag{3.24}$$

the second of which, in conjunction with (3.19), leads to

$$\left. \begin{array}{l} \bar{G}'_{11} = \overset{+}{G}'_{11} + 2\varkappa \overset{+}{G}'_{12} + \varkappa^2 \overset{+}{G}'_{22}, \quad \bar{G}'_{22} = \delta^2 \overset{+}{G}'_{22}, \\[2mm] \bar{G}'_{12} = \bar{G}'_{21} = \delta(\overset{+}{G}'_{12} + \varkappa \overset{+}{G}'_{22}). \end{array} \right\} \tag{3.25}$$

As for the physical interpretation of the parameters $\varkappa$ and δ, we infer first from (3.15), (3.19), (3.20) that

$$\bar{F} = \overset{+}{F}, \quad B = B' = 1 \text{ if and only if } \varkappa = 0, \ \delta = 1, \tag{3.26}$$

in which case the supplementary deformation (3.17) is the identity mapping and the original deformation (3.2) is trivial in the sense of being

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[1] Recall (1.14).

homogeneous on $\mathcal{R}$, and thus no longer constitutes an equilibrium shock. Moreover, (3.23) and the principle of mass conservation imply

$$\delta = \frac{\bar{J}}{\overset{+}{J}} = \frac{\overset{+}{\rho}}{\bar{\rho}} > 0 , \qquad (3.27)$$

where $\overset{+}{\rho}$ and $\bar{\rho}$ are the mass densities of the material occupying $\overset{+}{\mathcal{R}}_*$ and $\bar{\mathcal{R}}_*$ in the deformed configuration.   Thus (3.27) supplies an interpretation of δ in terms of the area-ratios or the ratio of mass densities appropriate to an equilibrium shock.

An additional kinematic meaning of δ, as well as a geometric interpretation of the parameter $\varkappa$, come into evidence if one factors the matrix $\underset{\sim}{B}'$ of (3.19) as follows:

$$\underset{\sim}{B}' = \begin{bmatrix} 1 & \varkappa \\ 0 & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} 1 & \varkappa \\ 0 & 1 \end{bmatrix} . \qquad (3.28)$$

This factorization reveals that the supplementary deformation (3.17) admits a decomposition into a simple shear of amount $\varkappa$, parallel to the z_1'-axis (spatial shock-line), followed by a uni-axial stretch, with the stretch-ratio δ, at right angles to this axis.   The preceding resolution is illustrated in Figure 6: the diagram in Figure 6(b) depicts the supplementary-deformation history of the two unit squares shown in Figure 6(a), which pertains to the intermediate configuration.   It should be emphasized that Figure 6 is based on $\delta < 1, \varkappa > 0$ and requires obvious modifications if $\delta \geq 1$ or $\varkappa \leq 0$.

For future purposes we mention here the important special case in which the deformation on $\overset{+}{\mathcal{R}}$ in (3.2) is <u>pure</u> homogeneous, having X as a principal frame. If λ_1, λ_2 are once again the corresponding principal stretches, one has in this particular instance

$$\overset{+}{\underset{\sim}{F}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \; , \; \overset{+}{\underset{\sim}{G}} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \, , \tag{3.29}$$

while (3.14) reduces to

$$\tan\varphi = \frac{\lambda_2}{\lambda_1} \tan\Phi. \tag{3.30}$$

We note also that the <u>second</u> of (3.29) holds true for an arbitrary homogeneous deformation on $\overset{+}{\underset{\sim}{R}}$, provided X is a principal frame for its deformation tensor $\overset{+}{\underset{\sim}{G}}$.

We return now to the <u>traction</u> <u>continuity</u> <u>condition</u> (3.7), which — on account of (3.3) — is equivalent to

$$W_{\underset{\sim}{F}} (\overset{+}{\underset{\sim}{F}}) \underset{\sim}{N} = W_{\underset{\sim}{F}} (\bar{\underset{\sim}{F}}) \underset{\sim}{N}. \tag{3.31}$$

Suppose $\overset{+}{\underset{\sim}{\tau}}$ and $\bar{\underset{\sim}{\tau}}$ represent the constant Cauchy (actual) stress field to which the piecewise homogeneous deformation (3.2) gives rise on $\overset{+}{\underset{\sim}{R}}_*$ and $\bar{\underset{\sim}{R}}_*$, respectively. Then, by (1.4) and (3.3),

$$\overset{+}{\underset{\sim}{\tau}} = \frac{1}{\overset{+}{J}} W_{\underset{\sim}{F}} (\overset{+}{\underset{\sim}{F}}) \overset{+}{\underset{\sim}{F}}^T \text{ on } \overset{+}{\underset{\sim}{R}}_* \; , \; \bar{\underset{\sim}{\tau}} = \frac{1}{\bar{J}} W_{\underset{\sim}{F}} (\bar{\underset{\sim}{F}}) \bar{\underset{\sim}{F}}^T \text{ on } \bar{\underset{\sim}{R}}_*. \tag{3.32}$$

An appeal to (3.11), the validity of which depends on the continuity of the displacements, at once confirms that (3.31), (3.32) imply

$$\overset{+}{\underset{\sim}{\tau}} \underset{\sim}{n} = \bar{\underset{\sim}{\tau}} \underset{\sim}{n} \equiv \underset{\sim}{t}, \tag{3.33}$$

where $\underset{\sim}{t}$ denotes the Cauchy (actual) traction exerted by the material occupying $\overset{+}{\underset{\sim}{R}}_*$ in the deformed configuration on the material in $\bar{\underset{\sim}{R}}_*$. Conversely, (3.33), (3.32), (3.11) assure that (3.31), and hence (3.7), holds true.

Thus, in the presence of continuous displacements, the nominal tractions are continuous across the material shock-line $\mathcal{L}$ if and only if the actual tractions are continuous across the spatial shock-line $\mathcal{L}_*$.

For later convenience we cite at this place also the particular form assumed by the traction continuity condition (3.33) in the event the hyper-elastic material under consideration happens to be isotropic. In this case (3.33), because of (1.18), furnishes

$$\left[\frac{2}{\overset{+}{J}}W_I(\overset{+}{I},\overset{+}{J})\overset{+}{\underset{\sim}{G}}+W_J(\overset{+}{I},\overset{+}{J})\underset{\sim}{1}\right]\underset{\sim}{n}=\left[\frac{2}{\overset{-}{J}}W_I(\overset{-}{I},\overset{-}{J})\overset{-}{\underset{\sim}{G}}+W_J(\overset{-}{I},\overset{-}{J})\underset{\sim}{1}\right]\underset{\sim}{n}, \qquad (3.34)$$

provided $\overset{+}{I},\overset{+}{J}$ and $\overset{-}{I},\overset{-}{J}$ stand for the scalar invariants of $\overset{+}{\underset{\sim}{G}}$ and $\overset{-}{\underset{\sim}{G}}$, so that in accordance with (1.14),

$$\overset{+}{I}=\operatorname{tr}\overset{+}{\underset{\sim}{G}}=\lambda_1^2+\lambda_2^2\ ,\ \ \overset{+}{J}=\sqrt{\det\overset{+}{\underset{\sim}{G}}}=\det\overset{+}{\underset{\sim}{F}}=\lambda_1\lambda_2,$$

$$\left.\begin{array}{c}\\ \\ \overset{-}{I}=\operatorname{tr}\overset{-}{\underset{\sim}{G}}\ ,\ \ \overset{-}{J}=\sqrt{\det\overset{-}{\underset{\sim}{G}}}=\det\overset{-}{\underset{\sim}{F}}.\end{array}\right\} \qquad (3.35)$$

It is clear from the second of (3.15), together with (3.9) and (3.21), that

$$\overset{-}{\underset{\sim}{F}}=\overset{-}{\underset{\sim}{F}}(\varphi,\varkappa,\delta)=\underset{\sim}{B}(\varphi,\varkappa,\delta)\overset{+}{\underset{\sim}{F}}. \qquad (3.36)$$

Hence, bearing in mind the developments leading up to (3.21), one sees that the displacement continuity condition (3.5) alone constrains the possible values of $\overset{-}{\underset{\sim}{F}}$, in a shock corresponding to a fixed prescribed $\overset{+}{\underset{\sim}{F}}$, to a three-parameter family. The existence of such a shock therefore hinges on the existence of a spatial shock-angle φ in $[-\pi/2,\pi/2]$, a real value of the shear parameter $\varkappa$, and a positive value of the stretch parameter δ, such that the traction continuity condition (3.33) has a solution $\overset{-}{\underset{\sim}{F}}(\varphi,\varkappa,\delta)$

other than the trivial solution supplied by

$$\bar{\underset{\sim}{F}} = \bar{\underset{\sim}{F}}(\varphi, 0, 1) = \underset{\sim}{B}(\varphi, 0, 1)\overset{+}{\underset{\sim}{F}} = \overset{+}{\underset{\sim}{F}} \quad (-\pi/2 \le \varphi \le \pi/2). \tag{3.37}$$

Furthermore, since (3.33) constitute only <u>two</u> scalar restrictions on the <u>three</u> parameters $(\varphi, \varkappa, \delta)$, one would anticipate that <u>if</u> there exists an equilibrium shock for a given deformation gradient $\overset{+}{\underset{\sim}{F}}$, there exists a one-parameter family of such shocks.

We now prove the following theorem, which establishes a <u>necessary</u> condition for the existence of shocks of the kind under consideration:

<u>If</u> <u>there</u> <u>exists</u> <u>a</u> <u>piecewise</u> <u>homogeneous</u> <u>elastostatic</u> <u>shock</u> <u>in</u> <u>a</u> <u>hyper-elastic</u> <u>material</u>, <u>then</u> <u>the</u> <u>displacement</u> <u>equations</u> <u>of</u> <u>equilibrium</u> <u>associated</u> <u>with</u> <u>this</u> <u>material</u> <u>must</u> <u>suffer</u> <u>a</u> <u>loss</u> <u>of</u> <u>strong</u> <u>ellipticity</u> <u>at</u> <u>some</u> <u>homogene-ous</u> <u>deformation</u>.

With a view toward establishing this claim, let $\overset{+}{\underset{\sim}{F}}$ and $\bar{\underset{\sim}{F}}$ be the deformation gradients of the existing shock, so that from (3.2), (3.5), and (3.31),

$$\det\overset{+}{\underset{\sim}{F}} > 0 \,, \quad \det\bar{\underset{\sim}{F}} > 0 \,, \quad \overset{+}{\underset{\sim}{F}}\underset{\sim}{L} = \bar{\underset{\sim}{F}}\underset{\sim}{L} \,, \quad W_{\underset{\sim}{F}}(\overset{+}{\underset{\sim}{F}})\underset{\sim}{N} = W_{\underset{\sim}{F}}(\bar{\underset{\sim}{F}})\underset{\sim}{N}, \tag{3.38}$$

where $\underset{\sim}{L}$ and $\underset{\sim}{N}$ are the unit direction and the unit normal vector of the material shock-line $\mathcal{L}$. Next define a family of tensors by means of

$$\underset{\sim}{F}(\alpha) = \alpha\overset{+}{\underset{\sim}{F}} + (1-\alpha)\bar{\underset{\sim}{F}} \quad (0 \le \alpha \le 1). \tag{3.39}$$

Then,

$$\underset{\sim}{F}(0) = \bar{\underset{\sim}{F}} \,, \quad \underset{\sim}{F}(1) = \overset{+}{\underset{\sim}{F}}, \tag{3.40}$$

while (3.39) and (3.15) give

$$\underset{\sim}{F}(\alpha) = [\alpha\underset{\sim}{1} + (1-\alpha)\underset{\sim}{B}]\overset{+}{\underset{\sim}{F}} \quad (0 \le \alpha \le 1). \tag{3.41}$$

From this relation, in turn, because of (3.20), (3.19), (3.27), and the first of (3.38), follows

$$\det \underset{\sim}{F}(\alpha) = (\alpha\delta + 1 - \alpha)\det \overset{+}{\underset{\sim}{F}} > 0 \quad (0 \le \alpha \le 1), \tag{3.42}$$

whence $\underset{\sim}{F}(\alpha)$ is admissible as a deformation-gradient tensor of a homogeneous plane deformation for every α in the interval $[0, 1]$.

According to the third of (3.38), there is a vector $\underset{\sim}{K}$ such that

$$\overset{-}{F}_{\gamma\delta} - \overset{+}{F}_{\gamma\delta} = K_\gamma N_\delta \ , \quad \underset{\sim}{K} \ne \underset{\sim}{0}. \tag{3.43}$$

Let Θ be the scalar-valued function defined by

$$\Theta(\alpha) = \underset{\sim}{K} \cdot [W_{\underset{\sim}{F}}(\underset{\sim}{F}(\alpha)) - W_{\underset{\sim}{F}}(\overset{+}{\underset{\sim}{F}})] \underset{\sim}{N} \quad (0 \le \alpha \le 1), \tag{3.44}$$

so that owing to (3.38) and (3.40),

$$\Theta(0) = \Theta(1) = 0. \tag{3.45}$$

Thus, by virtue of the mean-value theorem, there is a number $\hat{\alpha}$ in $(0, 1)$ such that

$$\left. \frac{d\Theta}{d\alpha} \right|_{\alpha = \hat{\alpha}} = 0. \tag{3.46}$$

But (3.44), (3.46) in conjunction with (3.39), (3.43), and (1.8) lead to

$$c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}(\hat{\alpha}))N_\beta N_\delta K_\alpha K_\gamma = 0. \tag{3.47}$$

Finally, set

$$\overset{\wedge}{\underset{\sim}{F}} = \underset{\sim}{F}(\hat{\alpha}) \ , \quad \underset{\sim}{M} = \underset{\sim}{K} / |\underset{\sim}{K}| \tag{3.48}$$

and appeal to (1.9) to arrive at

$$\underset{\sim}{M} \cdot \underset{\sim}{Q}(\underset{\sim}{N}; \hat{\underset{\sim}{F}}) \underset{\sim}{M} = 0, \tag{3.49}$$

which contradicts the strong-ellipticity condition (1.12). This completes the proof.

It remains a matter of conjecture as to whether or not the existence of an equilibrium shock necessitates not only a loss of strong ellipticity but also a failure of ordinary ellipticity, as is intuitively plausible. In this connection we refer once more to a result obtained in [2] and cited in Section 2 of the present paper.[1] It follows from this result that for an isotropic hyperelastic material a failure of strong ellipticity implies a failure of ordinary ellipticity, if the set of all points in the principal-stretch plane at which strong ellipticity prevails is a domain, i.e. open and connected, and includes the undeformed state.[2]

A related comment pertains to the existence of a piecewise homogeneous equilibrium shock within the linear theory of homogeneous and isotropic elastic solids. One verifies easily that in this setting such a shock exists if and only if the elastic constants satisfy

$$\overset{\bullet}{\mu} \neq 0 \text{ and } \overset{\bullet}{\lambda} + 2\overset{\bullet}{\mu} = 0 \text{ or } \overset{\circ}{\nu} = 1, \tag{3.50}$$

which require a failure of ordinary ellipticity of the linearized displacement equations of equilibrium.

4. Weak piecewise homogeneous elastostatic shocks.

Taking for granted the existence of piecewise homogeneous equilibrium shocks in the (possibly anisotropic) hyperelastic material under

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[1] See the discussion following (1.37).

[2] We recall from Section 2 that these conditions are met for the two special isotropic materials discussed there. See also Figure 2 and Figure 4.

consideration, we confine our attention in this section to shocks that are weak in the sense that $\underset{\sim}{\overset{-}{F}}$ remains close to $\underset{\sim}{\overset{+}{F}}$. Indeed, motivated by (3.36), (3.37), and the observations following (3.37), we assume here that there exists a one-parameter family of shocks corresponding to the given gradient $\underset{\sim}{\overset{+}{F}}$, depending on the parameter

$$\epsilon = 1 - \delta, \tag{4.1}$$

and suitably smooth near $\epsilon = 0$.

Specifically, we presuppose that there are functions $\varphi(\epsilon)$ and $\varkappa(\epsilon)$, both twice continuously differentiable in a neighborhood of $\epsilon = 0$, such that $\underset{\sim}{\overset{-}{F}}(\epsilon)$ defined by

$$\underset{\sim}{\overset{-}{F}}(\epsilon) = \underset{\sim}{B}(\varphi(\epsilon), \varkappa(\epsilon), \delta(\epsilon)) \underset{\sim}{\overset{+}{F}} \equiv \underset{\sim}{B}(\epsilon) \underset{\sim}{\overset{+}{F}}, \tag{4.2}$$

where

$$\delta(\epsilon) = 1 - \epsilon \ , \ \varkappa(0) = 0 \tag{4.3}$$

and $\underset{\sim}{B}(\varphi, \varkappa, \delta)$ is the tensor characterized by (3.21), conforms to the traction continuity condition (3.33) — or, equivalently, to (3.31) — throughout the neighborhood at hand. Since at present the trivial (shockless) solution (3.37) corresponds to $\epsilon = 0$ and ϵ is evidently a measure of the departure from this solution, we shall henceforth refer to ϵ as the shock-strength parameter. The kinematic significance of ϵ in terms of the supplementary deformation is immediate from that of δ (see Figure 6); also, (4.1) and (3.27) yield

$$\epsilon = \frac{\overset{+}{J} - \overset{-}{J}}{\overset{+}{J}} = \frac{\overset{-}{\rho} - \overset{+}{\rho}}{\overset{-}{\rho}}. \tag{4.4}$$

Our current objective is to explore various implications of the

existence of the ϵ-family of shocks postulated above, to the lowest signifi-
cant order in ϵ. For this purpose we observe first that the material shock-
line and the spatial shock-line are now ϵ-dependent. We thus write $\mathcal{L}(\epsilon), \mathcal{L}_*(\epsilon)$
in place of $\mathcal{L}, \mathcal{L}_*$ and consequently also $\underset{\sim}{L}(\epsilon), \underset{\sim}{N}(\epsilon), \underset{\sim}{\ell}(\epsilon), \underset{\sim}{n}(\epsilon)$, as well as $\overset{+}{\mathcal{R}}(\epsilon)$,
$\bar{\mathcal{R}}(\epsilon), \overset{+}{\mathcal{R}}_*(\epsilon), \bar{\mathcal{R}}_*(\epsilon)$ and $\Phi(\epsilon)$. Furthermore, any function previously defined on $\bar{\mathcal{R}}$
or $\bar{\mathcal{R}}_*$ and dependent on the value of the gradient $\underset{\sim}{F}$ is at present to be regarded
as a function of ϵ. With this understanding all of the results in Section 3 up
through (3.35) hold true identically in ϵ.

In view of the assumed smoothness of $\varphi(\epsilon), \varkappa(\epsilon)$ and by virtue of the
second of (4.3), one has the Taylor expansions

$$\underset{\sim}{N}(\epsilon) = \underset{\sim}{N}(0) + \underset{\sim}{\dot{N}}(0)\epsilon + o(\epsilon) \ , \ \underset{\sim}{n}(\epsilon) = \underset{\sim}{n}(0) + \underset{\sim}{\dot{n}}(0)\epsilon + o(\epsilon),$$

$$\varkappa(\epsilon) = \dot{\varkappa}(0)\epsilon + o(\epsilon) \text{ as } \epsilon \to 0; \tag{4.5}$$

here and in what follows a dot placed above a letter indicates differentiation
with respect to the parameter ϵ. On the other hand, using (3.21), (3.9), as
well as (4.3), one finds that $\underset{\sim}{B}(\epsilon)$ obeys

$$B_{\alpha\beta}(0) = \delta_{\alpha\beta} \ , \ \dot{B}_{\alpha\beta}(0) = q_\alpha n_\beta(0), \tag{4.6}$$

provided $\underset{\sim}{q}$ is the vector defined by

$$\underset{\sim}{q} = \dot{\varkappa}(0)\underset{\sim}{\ell}(0) - \underset{\sim}{n}(0). \tag{4.7}$$

From (4.6) follows

$$B_{\alpha\beta}(\epsilon) = \delta_{\alpha\beta} + q_\alpha n_\beta(0)\epsilon + o(\epsilon) \tag{4.8}$$

and, keeping (4.2) in mind, one thus draws

$$\bar{F}_{\alpha\beta}(\epsilon) = \overset{+}{F}_{\alpha\beta} + q_\alpha n_\gamma(0)\overset{+}{F}_{\gamma\beta}\epsilon + o(\epsilon). \tag{4.9}$$

Consequently, an appropriate two-term Taylor expansion leads to

$$\frac{\partial W(\bar{\underset{\sim}{F}}(\epsilon))}{\partial F_{\alpha\gamma}} = \frac{\partial W(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\gamma}} + \frac{\partial^2 W(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\gamma}\partial F_{\lambda\mu}} q_\lambda n_\rho(0)\overset{+}{F}_{\rho\mu}\epsilon + o(\epsilon). \tag{4.10}$$

Finally, entering (3.31) with (4.10) and using the first of (4.5), as well as (3.11), one sees that the traction continuity condition is fulfilled up to the order of ϵ if and only if

$$\frac{\partial^2 W(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\gamma}\partial F_{\lambda\mu}} N_\gamma(0)N_\mu(0)q_\lambda = 0. \tag{4.11}$$

This equation, in turn, may be written as

$$\underset{\sim}{Q}(\underset{\sim}{N}(0);\overset{+}{\underset{\sim}{F}})\underset{\sim}{q} = \underset{\sim}{0}, \tag{4.12}$$

where $\underset{\sim}{Q}(\underset{\sim}{N};\underset{\sim}{F})$ is the symmetric tensor introduced through (1.8), (1.9).

Now $\underset{\sim}{q} \neq \underset{\sim}{0}$ according to (4.7). Therefore (4.12) implies that

$$\det\underset{\sim}{Q}(\underset{\sim}{N}(0);\overset{+}{\underset{\sim}{F}}) = 0. \tag{4.13}$$

Drawing on the discussion of (1.10), (1.11) in Section 1, one is thus led to the following conclusions:

(i) A necessary condition for the existence of a one-parameter family of equilibrium shocks (of the kind under present consideration) is that the displacement equations of equilibrium associated with the hyperelastic material suffer a loss of ordinary ellipticity at the given homogeneous deformation on $\overset{+}{R}$.

(ii) In the weak-shock limit, i.e. as $\epsilon \to 0$, the material shock-line $\mathcal{L}(\epsilon)$ and the spatial shock-line $\mathcal{L}_*(\epsilon)$, respectively, tend to a material and a spatial characteristic associated with the homogeneous deformation on $\overset{+}{R}$.

In comparing conclusion (i) with the theorem proved at the end of Section 3, we note that the present result yields a breakdown of <u>ordinary</u> ellipticity — rather than merely of <u>strong</u> ellipticity; on the other hand (i) presupposes the existence of an entire family of elastostatic shocks, while in the theorem referred to but a single such shock was required to exist.

Conclusion (ii) restricts the number of possible distinct limiting material (and spatial) shock-lines to the number of distinct characteristic directions admitted by (1.11), which cannot exceed four. In particular, if the material is isotropic and X is chosen to be a principal frame for $\overset{+}{\underset{\sim}{G}}$ and $\overset{+}{\underset{\sim}{\tau}}$, the inclination $\varphi(0)$ of the limiting spatial shock-line $\mathcal{L}_{*}(0)$ must be such that $\cos 2\varphi(0)$ is a real root in the interval $[-1, 1]$ of the quadratic equation (1.38) for $\cos 2\gamma$, with γ replaced by $\varphi(0)$. As an illustrative example consider the isotropic Material 1 discussed in Section 2 and suppose that $\overset{+}{\underset{\sim}{F}}$ induces an <u>incipient</u> failure of ellipticity. Then, as is clear from the remarks following (2.18), $\varphi(0) = \pm \pi/4$ so that $\mathcal{L}_{*}(0)$ necessarily coincides with a trajectory of the maximum actual shear stress of the constant stress field $\overset{+}{\underset{\sim}{\tau}}$ prevailing on $\overset{+}{\underset{\sim}{R}}$.

A kinematic interpretation of the vector $\underset{\sim}{q}$ introduced in (4.7) is readily arrived at. Let $\underset{\sim}{w}(\underset{\sim}{z}; \epsilon)$ be the family of supplementary displacements corresponding to the ϵ-family of equilibrium shocks at hand. Then, by (3.17) and (4.8),

$$\underset{\sim}{w}(\underset{\sim}{z}; \epsilon) = \underset{\sim}{0} \text{ on } \overset{+}{\underset{\sim}{R}}_{*}(\epsilon), \tag{4.14}$$

$$\underset{\sim}{w}(\underset{\sim}{z}; \epsilon) = [\underset{\sim}{B}(\epsilon) - \underset{\sim}{1}]\underset{\sim}{z} = [\underset{\sim}{n}(0) \cdot \underset{\sim}{z}]\underset{\sim}{q}\epsilon + o(\epsilon) \text{ on } \bar{\underset{\sim}{R}}_{*}(\epsilon). \tag{4.15}$$

From (4.15) and the second of (4.5) one infers that

$$\nabla \underset{\sim}{w}(\underset{\sim}{z};\epsilon)\underset{\sim}{n}(\epsilon)=\underset{\sim}{q}\epsilon+o(\epsilon), \quad , \quad \nabla \underset{\sim}{w}(\underset{\sim}{z};\epsilon)\underset{\sim}{\ell}(\epsilon)=\underset{\sim}{0} \text{ on } \bar{R}_*(\epsilon), \tag{4.16}$$

provided $\nabla \underset{\sim}{w}(\underset{\sim}{z};\epsilon)$ is the tensor field with the components $\partial w_\alpha(\underset{\sim}{z};\epsilon)/\partial z_\beta$ in the frame X. For future convenience we now adopt the following notation: if f is a scalar or a vector field defined on $\overset{+}{R}$ and $\bar{R}$ (or on $\overset{+}{R}_*$ and $\bar{R}_*$) that suffers a finite jump discontinuity across $\mathcal{L}$ (or across $\mathcal{L}_*$), we write

$$[f]_+^- = \bar{f} - \overset{+}{f} \tag{4.17}$$

for the jump in f as $\mathcal{L}$ (or $\mathcal{L}_*$) is traversed from $\overset{+}{R}$ to $\bar{R}$ (or from $\overset{+}{R}_*$ to $\bar{R}_*$). Equations (4.16), (4.14) thus justify the assertions

$$[\nabla \underset{\sim}{w}\underset{\sim}{n}]_+^- = \underset{\sim}{q}\epsilon+o(\epsilon) \ , \quad [\nabla \underset{\sim}{w}\underset{\sim}{\ell}]_+^- = \underset{\sim}{0}. \tag{4.18}$$

Hence $q_\alpha \epsilon$ is the lowest-order approximation to the jump across $\mathcal{L}_*(\epsilon)$ in the derivative of $w_\alpha(\underset{\sim}{z};\epsilon)$ _normal_ to $\mathcal{L}_*(\epsilon)$, whereas the directional derivative of $w_\alpha(\underset{\sim}{z};\epsilon)$ _parallel_ to $\mathcal{L}_*(\epsilon)$ is continuous at $\mathcal{L}_*(\epsilon)$.

While the characteristic lines of the deformation on $\overset{+}{R}$ that emerge upon a loss of ordinary ellipticity are associated with jumps in the _second_ normal derivatives of the displacements[1], equilibrium shocks are seen, on the basis of (4.18), to involve discontinuities in the _first_ normal derivatives of the displacements across the shock-line. Also, it should be emphasized that $\mathcal{L}_*(\epsilon)$ is in general not a characteristic line of the deformation on $\overset{+}{R}(\epsilon)$, although it tends to such a line as $\epsilon \to 0$, i.e. in the weak-shock limit.

According to (4.1), (4.2), and (3.23), the jumps across $\mathcal{L}(\epsilon)$ in the area-ratio and the mass density obey

$$[J]_+^- = -\overset{+}{J}\epsilon = -\lambda_1 \lambda_2 \epsilon \ , \quad [\rho]_+^- = \bar{\rho}\epsilon. \tag{4.19}$$

---

[1] Recall the role of the characteristics reviewed in connection with (1.11).

In order to compute the jump across $\mathcal{L}(\epsilon)$ in the strain-energy density, we first rely on (4.9) to see that

$$W(\bar{\underset{\sim}{F}}(\epsilon)) = W(\overset{+}{\underset{\sim}{F}}) + \frac{\partial W(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} q_\alpha n_\gamma(0) F_{\gamma\beta} \epsilon + o(\epsilon). \qquad (4.20)$$

On account of (1.4) and (1.6), this equation may be written as

$$W(\bar{\underset{\sim}{F}}(\epsilon)) = W(\overset{+}{\underset{\sim}{F}}) + J\overset{+}{\underset{\sim}{q}} \cdot \overset{+}{\underset{\sim}{\tau}n}(0)\epsilon + o(\epsilon) \qquad (4.21)$$

or, by virtue of (3.33) and (4.17),

$$[W]_+^- = J\overset{+}{\underset{\sim}{t}}(0) \cdot \underset{\sim}{q}\epsilon + o(\epsilon) \ , \ \underset{\sim}{t}(0) = \overset{+}{\underset{\sim}{\tau}n}(0). \qquad (4.22)$$

We calculate next the jump across $\mathcal{L}_*(\epsilon)$ in the Cauchy stresses $\tau_{\alpha\beta}$. From (3.32), (4.19) follows

$$\bar{\tau}_{\alpha\beta}(\epsilon) = \frac{1}{\bar{J}(\epsilon)} \frac{\partial W(\bar{\underset{\sim}{F}}(\epsilon))}{\partial F_{\alpha\gamma}} \bar{F}_{\beta\gamma}(\epsilon) \ , \ \bar{J}(\epsilon) = (1-\epsilon)\overset{+}{J}. \qquad (4.23)$$

These formulas, together with (4.9), (4.10), and (1.8), permit one to deduce

$$[\tau_{\alpha\beta}]_-^+ = \left\{ \overset{+}{\tau}_{\alpha\beta} + \tau_{\alpha\rho} n_\rho(0) q_\beta + \frac{1}{\overset{+}{J}} c_{\alpha\gamma\lambda\mu}(\overset{+}{\underset{\sim}{F}}) \overset{+}{F}_{\rho\mu} \overset{+}{F}_{\beta\gamma} n_\rho(0) q_\lambda \right\} \epsilon + o(\epsilon). \qquad (4.24)$$

Finally, it is of interest to compute the jump across $\mathcal{L}_*(\epsilon)$ of the scalar normal stress acting parallel to the spatial shock-line, i.e. of τ'_{11}, if $\tau'_{\alpha\beta}$ are the components of $\underset{\sim}{\tau}$ when the latter is decomposed in the frame $\mathbf{X}'$ (see Figure 5). Clearly,

$$[\tau'_{11}]_+^- = [\tau_{\alpha\beta}]_+^- \ell_\alpha(\epsilon)\ell_\beta(\epsilon) \qquad (4.25)$$

and (3.10) yield

$$\ell_\alpha(\epsilon)\ell_\beta(\epsilon) = \ell_\alpha(0)\ell_\beta(0) + o(1) = \delta_{\alpha\beta} - n_\alpha(0)n_\beta(0) + o(1). \tag{4.26}$$

Substitution from (4.24) and (4.26) into (4.25), after a lengthy calculation that makes use of (4.7), (3.11), (1.9) and the traction continuity condition in the form (4.12), eventually leads to the result:

$$[\tau'_{11}]_+^- = \underset{\sim}{q} \cdot \overset{+}{\underset{\sim}{w}} \underset{\sim}{n}(0)\epsilon + o(\epsilon), \tag{4.27}$$

where $\overset{+}{\underset{\sim}{w}} = \underset{\sim}{w}(\overset{+}{\underset{\sim}{F}})$, while $\underset{\sim}{w}(\underset{\sim}{F})$ is the tensor with the components

$$w_{\alpha\beta}(\underset{\sim}{F}) = \tau_{\alpha\beta}(\underset{\sim}{F}) - \delta_{\alpha\beta}\tau_{\gamma\gamma}(\underset{\sim}{F}) + \frac{1}{J}c_{\lambda\mu\alpha\nu}(\underset{\sim}{F})F_{\lambda\mu}F_{\beta\nu} \tag{4.28}$$

in the frame X and $\underset{\sim}{\tau}(\underset{\sim}{F})$ the actual stress tensor associated with $\underset{\sim}{F}$ through (1.4), (1.6).

The lowest-order jump approximations (4.18), (4.22), (4.24), and (4.27) involve — beyond quantities fully determinable from the given $\overset{+}{\underset{\sim}{F}}$ and the known response function $W(\underset{\sim}{F})$ — the vector $\underset{\sim}{q}$ originally introduced in (4.7). Furthermore, this vector involves, in addition to the unit direction vector $\underset{\sim}{\ell}(0)$ and the unit normal vector $\underset{\sim}{n}(0)$ of the spatial characteristic $\mathcal{L}(0)$, also the still unknown value $\dot{\varkappa}(0)$.

We now determine $\dot{\varkappa}(0)$ from (4.12), and for this purpose exclude the degenerate case in which $\underset{\sim}{Q}(\underset{\sim}{N}(0);\overset{+}{\underset{\sim}{F}})$ is the null tensor. Thus assuming

$$\underset{\sim}{Q} \equiv \underset{\sim}{Q}(\underset{\sim}{N}(0);\underset{\sim}{F}) \neq \underset{\sim}{0}, \tag{4.29}$$

we show first that

$$\dot{\varkappa}(0) = 0 \text{ if and only if } \underset{\sim}{Q}\underset{\sim}{n}(0) = \underset{\sim}{0}. \tag{4.30}$$

To see this, suppose $\dot{\varkappa}(0)=0$.   Then $\underset{\sim}{Q}\underset{\sim}{n}(0)=\underset{\sim}{0}$ follows at once from (4.7),
(4.12).   Next, if $\dot{\varkappa}(0)\neq0$, $\underset{\sim}{Q}\underset{\sim}{n}(0)$ cannot vanish either since then $\underset{\sim}{q}$ and $\underset{\sim}{n}(0)$
would be linearly independent null vectors of the tensor $\underset{\sim}{Q}$, so that $\underset{\sim}{Q}$ would
have to be the null tensor, contrary to (4.29).   Hence (4.30) is true.   One
confirms similarly that

$$\underset{\sim}{n}(0)\cdot\underset{\sim}{Q}\underset{\sim}{\ell}(0)\neq0 \text{ if } \underset{\sim}{Q}\underset{\sim}{n}(0)\neq\underset{\sim}{0}. \tag{4.31}$$

Now (4.7), (4.12), because of (4.30), (4.31), furnish

$$\dot{\varkappa}(0)=0 \text{ if } \underset{\sim}{Q}\underset{\sim}{n}(0)=\underset{\sim}{0} \ , \ \dot{\varkappa}(0)=\frac{\underset{\sim}{n}(0)\cdot\underset{\sim}{Q}\underset{\sim}{n}(0)}{\underset{\sim}{n}(0)\cdot\underset{\sim}{Q}\underset{\sim}{\ell}(0)} \text{ if } \underset{\sim}{Q}\underset{\sim}{n}(0)\neq\underset{\sim}{0}, \tag{4.32}$$

where $\underset{\sim}{Q}$ again abbreviates $\underset{\sim}{Q}(\underset{\sim}{N}(0);\underset{\sim}{F})$.   Moreover, (4.32) and the last of (4.5)
give

$$\varkappa(\epsilon)=\frac{\underset{\sim}{n}(0)\cdot\underset{\sim}{Q}\underset{\sim}{n}(0)}{\underset{\sim}{n}(0)\cdot\underset{\sim}{Q}\underset{\sim}{\ell}(0)}\epsilon+o(\epsilon) \text{ if } \underset{\sim}{Q}\underset{\sim}{n}(0)\neq\underset{\sim}{0} \tag{4.33}$$

as a lowest-order weak-shock approximation to the amount of shear inherent
in the supplementary deformation.

Our next task is to specialize some of the foregoing results for the
case of material <u>isotropy</u>.   In this instance one finds with the aid of (1.9)
and (1.20) that, for every unit vector $\underset{\sim}{N}$,

$$Q_{\alpha\gamma}(\underset{\sim}{N};\underset{\sim}{F})=2W_I\delta_{\alpha\gamma}+$$

$$4W_{II}F_{\alpha\beta}N_\beta F_{\gamma\rho}N_\rho+J^2W_{JJ}F^{-1}_{\beta\alpha}N_\beta F^{-1}_{\rho\gamma}N_\rho+$$

$$2JW_{IJ}(F^{-1}_{\beta\alpha}N_\beta F_{\gamma\rho}N_\rho+F^{-1}_{\beta\gamma}N_\beta F_{\alpha\rho}N_\rho). \tag{4.34}$$

We now use (4.34) in (3.66), invoke (3.11), (3.12), (3.13), as well as (3.9),

and choose $\mathbf{X}$ as a principal frame for $\overset{+}{\underset{\sim}{G}}$, so that the component matrix of $\overset{+}{\underset{\sim}{G}}$ in $\mathbf{X}$ is given by the second of (3.29). In this manner we eventually arrive at the following result for $\dot{\varkappa}(0)$, provided (4.29) holds and $\underset{\sim}{Q}(\underset{\sim}{N}(0);\overset{+}{\underset{\sim}{F}})\underset{\sim}{n}(0)$ also fails to vanish:

$$\dot{\varkappa}(0) = -\frac{2aW_I + 4a^2W_{II} + \overset{+}{J}{}^2 W_{JJ} + 4a\overset{+}{J}W_{IJ}}{2(2aW_{II} + \overset{+}{J}W_{IJ})b} , \tag{4.35}$$

where

$$a = \lambda_1^2 \sin^2\varphi(0) + \lambda_2^2 \cos^2\varphi(0) \ , \ \ b = (\lambda_1^2 - \lambda_2^2)\sin\varphi(0)\cos\varphi(0) \tag{4.36}$$

and the partial derivatives of W with respect to the deformation invariants are understood to be evaluated at $(\overset{+}{I},\overset{+}{J})$, the latter being supplied by (3.35).

If one adheres to the above special choice of the frame $\mathbf{X}$, (4.22) is in the present circumstances found to imply

$$[W]_+^- = -(2dW_I + \overset{+}{J}W_J)\epsilon + o(\epsilon), \tag{4.37}$$

provided

$$d = a + b\dot{\varkappa}(0) = -\frac{2aW_I + \overset{+}{J}{}^2 W_{JJ} + 2a\overset{+}{J}W_{IJ}}{2(2aW_{II} + \overset{+}{J}W_{IJ})} \tag{4.38}$$

and a is given by the first of (4.36). The tensor $\underset{\sim}{\omega}$ introduced in (4.28) in the case of the isotropic hyperelastic solid turns out to have the components

$$\omega_{\alpha\beta} = \frac{4}{J}(W_I + IW_{II} + JW_{IJ})G_{\alpha\beta} - \frac{2}{J}(IW_I - J^2 W_{JJ} - IJW_{IJ})\delta_{\alpha\beta}. \tag{4.39}$$

Finally, (4.27) and (4.39) enable one to confirm that

$$[\tau'_{11}]^-_+ = -\frac{2}{\overset{+}{J}}\left[2d\left(W_I + \overset{+}{I}W_{II} + \overset{+}{J}W_{IJ}\right) - \overset{+}{I}W_I + \overset{+}{J}{}^{2}W_{JJ} + \overset{+}{I}\overset{+}{J}W_{IJ}\right]\epsilon + o(\epsilon), \qquad (4.40)$$

if the previous choice of X is retained. Also, the derivatives of W appearing in (4.37), (4.38), (4.40) stand for their corresponding values at $(\overset{+}{I}, \overset{+}{J})$.

For future purposes we record here the weak-shock approximations of $\varkappa(\epsilon)$ for the two special isotropic materials discussed in Section 2. Recalling (2.1), (2.3), we note that the strain-energy densities governing the reponse of these materials obey:

$$W(I, J) = \frac{\mu}{2}(IJ^{-2} + 2J - 4) \quad (\mu > 0) \text{ for Material 1,}$$

$$W(I, J) = \frac{\mu}{2}(IJ + 8J^{-\frac{1}{2}} - 10) \quad (\mu > 0) \text{ for Material 2.}$$
$$\left.\rule{0pt}{42pt}\right\} \quad (4.41)$$

On the basis of (4.35), (4.41) and (4.5) one finds that

$$\varkappa(\epsilon) = \frac{3}{2b}(\overset{+}{I} - a)\epsilon + o(\epsilon) \text{ for Material 1,}$$

$$\varkappa(\epsilon) = -\frac{3}{b}(a + \overset{+}{J}{}^{-\frac{3}{2}})\epsilon + o(\epsilon) \text{ for Material 2,}$$
$$\left.\rule{0pt}{42pt}\right\} \quad (4.42)$$

with a and b given by (4.36).

In connection with the weak-shock jump estimates deduced in this section, it is essential to recognize that the _signs_ of some of these jumps may depend not only on the particular material considered, but also on the particular nature of the pre-assigned deformation on $\overset{+}{\mathcal{R}}$; moreover, the sign of _each_ jump depends on the sign of the shock-strength parameter ϵ. The sign of ϵ, in turn, cannot be determined in the absence of information beyond that contained in our present characterization of piecewise homogeneous elastostatic shocks. In Section 6 we shall, on energetic grounds,

propose an additional requirement that leads to a removal of this indeterminacy.

5. <u>Equilibrium shocks of finite strength for a particular isotropic hyper-elastic material</u>.

We turn now to an instructive illustrative example concerning the <u>global</u> existence and character of piecewise homogeneous elastostatic shocks in a special (homogeneous) isotropic hyperelastic material. The following analysis is based on <u>Material</u> 1 of Section 2, the strain-energy density of which is given by the first of (4.41). To simplify this analysis we shall assume here from the start that the second of (3.29) is in force, so that X is a principal frame for the deformation tensor $\overset{+}{\underset{\sim}{G}}$ of the given deformation on $\overset{+}{\underset{\sim}{R}}$, and hence for the actual stress tensor $\overset{+}{\underset{\sim}{\tau}}$ as well.

The traction continuity condition (3.34) at present reduces to

$$\frac{1}{\overset{+}{J}{}^3}(\overset{+}{\underset{\sim}{G}}-\overset{+}{\underset{\sim}{1}})\underset{\sim}{n}=\frac{1}{\bar{J}{}^3}(\overset{-}{\underset{\sim}{G}}-\overset{-}{\underset{\sim}{1}})\underset{\sim}{n}. \tag{5.1}$$

Upon referring (5.1) to the frame X' introduced in Section 3 (see Figure 5) and bearing the first of (3.35) in mind, one arrives at

$$\frac{\overset{+}{G}{}'_{12}}{\overset{+}{J}{}^3}=\frac{\bar{G}'_{12}}{\bar{J}{}^3}\,,\quad \frac{\overset{+}{G}{}'_{11}}{\overset{+}{J}{}^3}=\frac{\bar{G}'_{11}}{\bar{J}{}^3}. \tag{5.2}$$

On setting

$$\beta=\frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}\quad(\lambda_1>0,\lambda_2>0), \tag{5.3}$$

one obtains by means of the second of (3.29) and the second of (3.20):

$$\overset{+}{G}{}'_{11} = \frac{1}{4}(\lambda_1 + \lambda_2)^2 (1 - 2\beta \cos 2\varphi + \beta^2),$$

$$\overset{+}{G}{}'_{22} = \frac{1}{4}(\lambda_1 + \lambda_2)^2 (1 + 2\beta \cos 2\varphi + \beta^2), \tag{5.4}$$

$$\overset{+}{G}{}'_{12} = \frac{1}{2}(\lambda_1 + \lambda_2)^2 \beta \sin 2\varphi.$$

The components $\bar{G}'_{\alpha\beta}$ are now computable from (3.25), while $\bar{J}$ is related to $\overset{+}{J}$ through the first of (3.27). In this manner (5.2) lead to the two scalar traction continuity conditions

$$2\delta^2 \beta \sin 2\varphi = \varkappa(1 + 2\beta \cos 2\varphi + \beta^2) + 2\beta \sin 2\varphi,$$

$$\delta^3 (1 - 2\beta \cos 2\varphi + \beta^2) = \tag{5.5}$$

$$1 - 2\beta \cos 2\varphi + \beta^2 + \varkappa^2 (1 + 2\beta \cos 2\varphi + \beta^2) + 4\varkappa\beta \sin 2\varphi.$$

Observe that the given deformation on $\overset{+}{\mathcal{R}}$, in the example under consideration, enters (5.5) exclusively through the principal-stretch parameter β adopted in (5.3). Also, (5.3) and (3.27) require that β and δ satisfy

$$-1 < \beta < 1 \ , \quad \delta \equiv 1 - \epsilon > 0, \tag{5.6}$$

where ϵ is the shock-strength parameter introduced in (4.1). From here on we shall give preference to ϵ over δ. The trivial solution of (5.5), which signifies the absence of a shock, is furnished by $\epsilon = 0$ ($\delta = 1$), $\varkappa = 0$, and $-\pi/2 \leq \varphi \leq \pi/2$. We now seek the answer to the following question. For what values of β in $(-1, 1)$ and $\epsilon < 1$ do there exist nontrivial solutions $\varkappa = \varkappa(\epsilon; \beta)$ and $\varphi = \varphi(\epsilon; \beta)$ of the simultaneous equations (5.5), such that $\varkappa$ is real and φ in the interval $[-\pi/2, \pi/2]$? Moreover, we wish to find all such solutions.

It is clear from the structure of (5.5) that if a pair of values $(\varkappa, \varphi)$

satisfies these two equations, then so does the pair $(-\varkappa, -\varphi)$. Consequently, the limitation of the spatial shock-angle to the range

$$0 \leq \varphi \leq \pi/2 \tag{5.7}$$

entails no loss in generality. From the first of (5.5) and the second of (5.6) one has

$$\varkappa = \frac{2(\delta^2-1)\beta \sin 2\varphi}{1+2\beta \cos 2\varphi + \beta^2}, \tag{5.8}$$

and substitution for $\varkappa$ from (5.8) into the second of (5.5), in view of (5.6) and (5.7), is found to yield

$$\sin 2\varphi = \frac{1-\beta^2}{2|\beta| k(\epsilon)} \; , \quad k(\epsilon) = \sqrt{\frac{\delta^3}{\sqrt{\delta^2+\delta+1}}} > 0 \; , \quad \delta = 1-\epsilon, \tag{5.9}$$

if once and for all we remove the indeterminacy in φ at $\epsilon = 0$ by requiring continuity. Since φ is to be real, the positive right member[1] of the first of (5.9) cannot exceed unity. Hence,

$$\frac{1-\beta^2}{2|\beta|} \leq k(\epsilon) \quad (-\infty < \epsilon < 1) \; , \quad |\beta| < 1. \tag{5.10}$$

These two inequalities are easily seen to be equivalent to

$$h(\epsilon) \leq |\beta| < 1 \; , \quad h(\epsilon) = [k(\epsilon) + \sqrt{k^2(\epsilon)+1}]^{-1} \quad (-\infty < \epsilon < 1) \tag{5.11}$$

and are necessary in order that (5.5) admit a solution $(\varkappa, \varphi)$, subject to (5.6), (5.7), for a given pair of values $(\epsilon \equiv 1-\delta, \beta)$. Conversely, if (5.11) is fulfilled, such a solution is supplied by (5.8), (5.9).

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[1] Recall (5.6).

Thus, in the present circumstances, an elastostatic shock of strength $\epsilon<1$ exists if and only if the principal stretches λ_1, λ_2 inherent in the pre-assigned deformation on $\overset{+}{\mathcal{R}}$, which determine the parameter β through (5.3), are such that the point with the rectangular cartesian coordinates $(\epsilon, |\beta|)$ of the $(\epsilon, |\beta|)$-plane lies in the "admissible region" G characterized by (5.11) and shown in Figure 7. In this connection we note on the basis of the definitions of $k(\epsilon)$ and $h(\epsilon)$ in (5.9), (5.11) that

$$\left.\begin{aligned} k_o \equiv k(0) = h(0) = \frac{1}{\sqrt{3}}, \\[2mm] k(1-) = 0 , \ h(1-) = 1 , \ h(-\infty) = 0 , \ \dot{k}(1-) = \dot{h}(1-) = 0, \\[2mm] \dot{k}(\epsilon) < 0 , \ \dot{h}(\epsilon) > 0 \ \ (-\infty < \epsilon < 1), \end{aligned}\right\} \quad (5.12)$$

provided primes denote differentiation with respect to the shock-strength parameter ϵ.

According to (5.9), every pair of values (ϵ, β) admitted by (5.10) or (5.11) gives rise to two spatial shock-angles in the interval $[0, \pi/2]$:

$$\left.\begin{aligned} \varphi_1(\epsilon; \beta) = \frac{1}{2} \sin^{-1}\left[\frac{1-\beta^2}{2|\beta| k(\epsilon)}\right] \ (0 \leq \varphi_1 \leq \pi/4) , \\[2mm] \varphi_2(\epsilon; \beta) = \frac{\pi}{2} - \varphi_1(\epsilon; \beta). \end{aligned}\right\} \quad (5.13)$$

The amounts of (supplementary) shear, κ_1 and κ_2, associated with the respective shock-angles φ_1 and φ_2 are readily found by recourse to (5.8) and (5.13). In this manner one obtains

$$\kappa_\alpha(\epsilon; \beta) = \frac{\delta^2 - 1}{R_\alpha(\epsilon; \beta)} \ (\alpha = 1, 2) , \ \delta = 1 - \epsilon, \quad (5.14)$$

where

$$R_\alpha(\epsilon;\beta) = \text{sgn}\beta \frac{1+\beta^2}{1-\beta^2} k(\epsilon) + (-1)^{\alpha+1} Q(\epsilon;\beta) \ , \quad Q(\epsilon;\beta) = \left[\frac{4\beta^2 k^2(\epsilon)}{(1-\beta^2)^2} - 1\right]^{\frac{1}{2}}. \tag{5.15}$$

We observe that $Q(\epsilon;\beta)$ is real whenever (ϵ, β) conform to the admissibility requirement (5.10) and that

$$\varphi_1(\epsilon;\beta) = \varphi_2(\epsilon;\beta) = \frac{\pi}{4} \ , \quad \varkappa_1(\epsilon;\beta) = \varkappa_2(\epsilon;\beta) \text{ if } \frac{1-\beta^2}{2|\beta|} = k(\epsilon), \tag{5.16}$$

i.e., if $(\epsilon, |\beta|)$ is a point on the boundary of the admissibility region G displayed in Figure 7. Further, as is apparent from (5.14), (5.15),

$$\varkappa_1(\epsilon;-\beta) = -\varkappa_2(\epsilon;\beta) \ , \quad \varkappa_2(\epsilon;-\beta) = -\varkappa_1(\epsilon;\beta). \tag{5.17}$$

With reference to the remark preceding (5.7) we emphasize that (5.13), (5.14) in general determine four distinct equilibrium shocks appropriate to the particular material at hand.

One confirms easily with the aid of (2.13), (3.35), and (5.4) that the displacement equations of equilibrium are elliptic at the homogeneous deformation given on $\overset{+}{\mathcal{R}}$ if and only if

$$|\beta| < k_o = \frac{1}{\sqrt{3}}. \tag{5.18}$$

For the sake of brevity we shall say that "ellipticity prevails on $\overset{+}{\mathcal{R}}$" whenever (5.18) holds true. Evidently, such is the case at all points $(\epsilon, |\beta|)$ of the admissible region G (Figure 7) that lie below the line $|\beta| = k_o$, while all points of G on or above that line correspond to a failure of ellipticity on $\overset{+}{\mathcal{R}}$. In particular, such a failure of ellipticity on $\overset{+}{\mathcal{R}}$ occurs for all admissible pairs $(\epsilon, |\beta|)$ with $\epsilon \geq 0$.

We now seek to ascertain for what points in G the ensuing homogeneous deformation on $\overset{-}{\mathcal{R}}$ is elliptic or otherwise. According to (2.13),

ellipticity prevails on $\bar{\mathcal{R}}$ if and only if

$$\bar{I} < 4\bar{J}. \tag{5.19}$$

On appealing to (3.35), (3.25), (3.23), (5.4), remembering that $\bar{\mathsf{G}}$ and $\bar{\mathsf{G}}'$ have the same trace, we find with the aid of (5.8) — after a rather lengthy computation — that

$$\bar{I} = \frac{(\lambda_1 + \lambda_2)^2}{4} \left[(1+\beta^2)(1+\delta^2) + 2\beta(\delta^2 - 1)\cos 2\varphi \right.$$

$$\left. + \frac{4\beta^2(\delta^4 - 1)\sin^2 2\varphi}{1 + 2\beta\cos 2\varphi + \beta^2} \right], \quad \bar{J} = \delta\lambda_1\lambda_2, \tag{5.20}$$

in which $\varphi = \varphi_\alpha(\epsilon;\beta)$ is furnished by (5.13). Owing to (5.20) and (5.3), the ellipticity condition (5.19) for $\bar{\mathcal{R}}$ becomes

$$\frac{1}{4\delta(1-\beta^2)} \left[(1+\beta^2)(1+\delta^2) + 2\beta(\delta^2-1)\cos 2\varphi + \frac{4\beta^2(\delta^4-1)\sin^2 2\varphi}{1+2\beta\cos 2\varphi + \beta^2} \right] < 0. \tag{5.21}$$

The inequality (5.21), together with (5.6), enables one to prove that there is a failure of ellipticity on $\bar{\mathcal{R}}$ for all admissible pairs $(\epsilon, |\beta|)$ with $\epsilon \leq 0$; in contrast, the subregion of G corresponding to $\epsilon > 0$ contains points $(\epsilon, |\beta|)$ at which ellipticity prevails on $\bar{\mathcal{R}}$, as well as points at which it does not.

For reasons that will become apparent later on, we confine our attention in the remainder of this section to shocks of strength $\epsilon \geq 0$.

Figure 8, which relies on (5.13), (5.14), illustrates the dependence of the two spatial shock-angles $\varphi_\alpha(\epsilon;\beta)$ within the interval $[0, \pi/2]$, and of the associated amounts of shear $\varkappa_\alpha(\epsilon;\beta)$, upon ϵ for $\epsilon \geq 0$ and $\beta = 0.65$. At this value of β the maximum possible ϵ is approximately 0.22, as is clear from Figure 7. Both $\varkappa_1$ and $\varkappa_2$ are negative[1] for $0 < \epsilon \leq \epsilon_{max}$. Observe that, in

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[1] The schematic inset diagram at the top of Figure 8 corresponds to a positive value of $\varkappa$.

in agreement with (5. 16), $\varphi_1 = \varphi_2 = \pi/4$ and $\varkappa_1 = \varkappa_2$ when $\epsilon = \epsilon_{max}$, in which case $(\epsilon, |\beta|)$ is a point on the boundary of the admissibility region G. While φ_1, φ_2, and $\varkappa_1$ vary monotonically with ϵ, the curve for $\varkappa_2$ in Figure 8 exhibits a minimum in the interior of the interval $[0, \epsilon_{max}]$.

The jump in the strain-energy density W, across the appropriate material shock-line, is deducible from the first of (4. 41) by means of (3. 35) and (5. 20). The result of this computation may be put into the form

$$[W]_+^- = \frac{\mu(\delta-1)}{2\delta^2 \lambda_1 \lambda_2} \left[\frac{(1-\beta^2)(1+\delta)(1+\delta^2)}{\delta(1+2\beta\cos2\varphi+\beta^2)} + 2\lambda_1^2\lambda_2^2\delta^2 \right], \qquad (5. 22)$$

where $\varphi = \varphi_\alpha(\epsilon;\beta)$ is available from (5. 13) and $\alpha = 1$ or $\alpha = 2$ depending on the particular shock at hand. Since $\delta = 1 - \epsilon > 0$, it follows that all shocks of positive strength ϵ possible in Material 1 give rise to a decrease in the energy density as the shock-line $\mathcal{L}$ is traversed from $\overset{+}{\mathcal{R}}$ to $\overset{-}{\mathcal{R}}$.

Because of (5. 12), (5. 13),

$$\varphi_1(0;\beta) = \frac{1}{2} \sin^{-1} \left[\frac{\sqrt{3}(1-\beta^2)}{2|\beta|} \right] \quad (0 \leq \varphi_1 \leq \pi/4), \quad \varphi_2(0;\beta) = \frac{\pi}{2} - \varphi_1(0;\beta). \quad (5. 23)$$

It is not difficult to verify by recourse to (5. 3) that equations (2. 18) are satisfied if γ equals either of the limiting spatial shock-angles in (5. 23). Hence, in the limit as $\epsilon \to 0$, the spatial shock-lines appropriate to Material 1 tend to the spatial characteristics associated with the prescribed homogeneous deformation on $\overset{+}{\mathcal{R}}$, the latter being accompanied by a loss of ellipticity. This conclusion reflects a general result concerning weak elastostatic shocks[1] established in Section 4. Similarly, a Taylor expansion about $\epsilon = 0$ applied to (5. 14) confirms the consistency of this global result with its weak-shock counterpart in (4. 42).

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[1] See the discussion of (4. 13).

6. <u>Time-dependent piecewise homogeneous elastostatic shocks</u>. <u>Energy</u>
   <u>considerations</u>. <u>Dissipativity</u>.

The loss of ellipticity of the field equations of elastostatics at par-
ticular — sufficiently severe — deformations of certain hyperelastic solids is
analogous to the change of type that may occur in the partial differential equa-
tions governing <u>steady</u> irrotational flows of a compressible, inviscid fluid[1].
These equations are elliptic or hyperbolic at a point of the flow field accord-
ing as the corresponding particle velocity is subsonic or supersonic.

One of the important features of compressible flows is the possible
occurrence of shock-surfaces across which the fluid pressure, density and
velocity, as well as the entropy, suffer jump discontinuities. The simplest
example of this kind in the theory of <u>steady</u> <u>plane</u> <u>flows</u> is that in which a
plane of discontinuity separates two uniform flows. Further, this discontin-
uous flow field is a close analogue of a piecewise homogeneous elastostatic
shock. The mathematical counterpart in the foregoing fluid-flow problem of
the conditions of displacement and traction continuity are restrictions aris-
ing from the balance of mass, momentum, and energy across the gas-dynam-
ical shock-plane. The ensuing one-parameter family of shocks[2] may be re-
ferred to the shock-strength parameter

$$\epsilon = \frac{\bar{\rho} - \overset{+}{\rho}}{\bar{\rho}}. \tag{6.1}$$

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[1] See Courant and Friedrichs [7] for a general treatment of compressible
flows. A detailed discussion of the strict analogy between steady gas flows
and anti-plane shear in finite elastostatics may be found in [8].

[2] Actually there are two symmetrically located families of shocks; in contrast,
the higher-order elasticity problem gives rise to two <u>pairs</u> of symmetrically
situated one-parameter families of equilibrium shocks — at least in the exam-
ple of Section 5.

where $\overset{+}{\rho}$ and $\overset{-}{\rho}$ are the respective fluid densities on the upstream and down-stream side of the shock. Equation (6.1) is identical with the second of (4.4), in which $\overset{+}{\rho}$ and $\overset{-}{\rho}$ are the mass densities of the elastic material occupy-ing $\overset{+}{\mathcal{R}_*}$ and $\overset{-}{\mathcal{R}_*}$, respectively, in the deformed state.

In gas dynamics the shock conditions mentioned above are accom-panied by the independent requirement that the entropy of a fluid particle shall increase as the shock-surface is traversed. This entropy condition models the dissipative nature of the process of shock-formation in the ab-sence of viscosity; it leads directly to the conclusion that ϵ must be positive and hence $\overset{-}{\rho} > \overset{+}{\rho}$.

The role of entropy-like conditions was studied extensively by Lax [9][1] for quasi-linear hyperbolic systems of conservation laws in two inde-pendent variables. Lax's work is motivated primarily by the initial-value problem for a system of partial differential equations in which time appears explicitly as one of the two independent variables.

In view of our remarks concerning stationary shocks in gas dynam-ics, as well as on independent physical grounds, it is natural to subject elas-tostatic shocks likewise to an additional limitation that assures their dissi-pative character. For this purpose we find it essential to generalize the no-tion of an elastostatic shock defined in Section 3 to a time-dependent family of such equilibrium shocks. It should be made clear that time will play merely the part of a history parameter in this context since there are no inertia ef-fects involved in the following quasi-static considerations.

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[1] This paper contains references to related earlier investigations by the same author.

As in Section 3, let $\Re$ stand for the entire (x_1, x_2)-plane spanned by a fixed (time-independent) rectangular cartesian coordinate frame X. Let $\mathcal{L}(t)$ $(t_1 \leq t \leq t_2)$ be a <u>time</u>-<u>dependent</u> family of straight lines, which conforms to the parametrization

$$\mathcal{L}(t): \underset{\sim}{x} = \underset{\sim}{\hat{x}}(\xi, t) = \underset{\sim}{p}(t) + \xi \underset{\sim}{L}(t) \quad (-\infty < \xi < \infty, t_1 \leq t \leq t_2). \tag{6.2}$$

Here $L(t)$ is the orienting unit direction vector of $\mathcal{L}(t)$ at the time t, while $\underset{\sim}{p}(t)$ is the instantaneous position vector of a point that is attached to $\mathcal{L}$ and participates in the rigid motion of $\mathcal{L}$ relative to the frame X; evidently, ξ is the directed distance from this point, measured along $\mathcal{L}$ (see Figure 9). We take for granted that $\underset{\sim}{p}$ and $\underset{\sim}{L}$ are continuously differentiable functions of the time on the interval $[t_1, t_2]$. Further, paralleling the agreements introduced at the beginning of Section 3, we call $\underset{\sim}{N}(t)$ the unit normal vector of $\mathcal{L}(t)$ resulting from a counter-clockwise rotation of $\underset{\sim}{L}(t)$ through $\pi/2$. Also, we denote by $\overset{+}{\Re}(t)$ and $\bar{\Re}(t)$ the open half-planes into which $\Re$ is divided by $\mathcal{L}(t)$ at the instant t, with the proviso that $\underset{\sim}{N}(t)$ points into $\overset{+}{\Re}(t)$.

At this stage we define a time-dependent, piecewise homogeneous, family of plane deformations through

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x}, t) = \begin{cases} \overset{+}{\underset{\sim}{F}}(t)\underset{\sim}{x} + \overset{+}{\underset{\sim}{b}}(t) & \text{for all } \underset{\sim}{x} \text{ in } \overset{+}{\Re}(t) \quad (t_1 \leq t \leq t_2) \\[2mm] \bar{\underset{\sim}{F}}(t)\underset{\sim}{x} + \bar{\underset{\sim}{b}}(t) & \text{for all } \underset{\sim}{x} \text{ in } \bar{\Re}(t) \quad (t_1 \leq t \leq t_2), \end{cases}$$

$$\overset{+}{J} = \det \overset{+}{\underset{\sim}{F}} > 0 \ , \ \bar{J} = \det \bar{\underset{\sim}{F}} > 0 \text{ on } [t_1, t_2]. \tag{6.3}$$ [1]

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[1] These equations are the time-dependent counterpart of (3.2). Note that the translation vectors $\overset{+}{\underset{\sim}{b}}, \bar{\underset{\sim}{b}}$ are needed here since $\mathcal{L}(t)$, given by (6.2), may not pass through the origin at all times. Actually, but for a lack of symmetry, one of these two vectors could have been omitted.

The (position-independent) deformation gradients $\overset{+}{\underset{\sim}{F}}, \overset{-}{\underset{\sim}{F}}$, as well as the translation vectors $\overset{+}{\underset{\sim}{b}}, \overset{-}{\underset{\sim}{b}}$, are required to possess continuous first time-derivatives on $[t_1, t_2]$.

Instead of (3.3) we now have

$$\overset{+}{\underset{\sim}{\sigma}}(t) = W_{\underset{\sim}{F}}(\overset{+}{\underset{\sim}{F}}(t)) \text{ on } \overset{+}{\mathbb{R}}(t) \ , \ \overset{-}{\underset{\sim}{\sigma}}(t) = W_{\underset{\sim}{F}}(\overset{-}{\underset{\sim}{F}}(t)) \text{ on } \overset{-}{\mathbb{R}}(t) \ (t_1 \leq t \leq t_2), \quad (6.4)$$

where $\overset{+}{\underset{\sim}{\sigma}}$ and $\overset{-}{\underset{\sim}{\sigma}}$ are the nominal stress fields produced by the time-dependent deformation (6.3) on $\overset{+}{\mathbb{R}}$ and $\overset{-}{\mathbb{R}}$, if the solid occupying $\mathbb{R}$ in its undeformed configuration is hyperelastic and W is its strain-energy density. The displacement-continuity condition (3.4) at present gives way to

$$\overset{+}{\underset{\sim}{F}}(t)\underset{\sim}{x} + \overset{+}{\underset{\sim}{b}}(t) = \overset{-}{\underset{\sim}{F}}(t)\underset{\sim}{x} + \overset{-}{\underset{\sim}{b}}(t) \text{ for all } \underset{\sim}{x} \text{ on } \mathcal{L}(t) \ (t_1 \leq t \leq t_2); \quad (6.5)$$

because of (6.3), this requirement is met if and only if

$$\overset{+}{\underset{\sim}{F}}\underset{\sim}{L} = \overset{-}{\underset{\sim}{F}}\underset{\sim}{L} \ , \ \overset{+}{\underset{\sim}{F}}\underset{\sim}{p} + \overset{+}{\underset{\sim}{b}} = \overset{-}{\underset{\sim}{F}}\underset{\sim}{p} + \overset{-}{\underset{\sim}{b}} \text{ on } [t_1, t_2], \quad (6.6)$$

which take the place of (3.5). Assuming the mapping (6.3) to have been extended continuously onto $\mathcal{L}(t)$, we let $\mathcal{L}_*(t)$ stand for the deformation image of $\mathcal{L}(t)$ ($t_1 \leq t \leq t_2$).

The stress field (6.4) clearly satisfies the equilibrium equation (1.5) on $\overset{+}{\mathbb{R}}(t)$ and $\overset{-}{\mathbb{R}}(t)$ at each instant; but the balance of force across $\mathcal{L}(t)$ demands the continuity of the nominal tractions[1] at $\mathcal{L}(t)$:

$$\overset{+}{\underset{\sim}{\sigma}}\underset{\sim}{N} = \overset{-}{\underset{\sim}{\sigma}}\underset{\sim}{N} \equiv \underset{\sim}{s} \text{ on } [t_1, t_2]. \quad (6.7)$$

When $\overset{+}{\underset{\sim}{F}}(t) \neq \overset{-}{\underset{\sim}{F}}(t)$ for $t_1 \leq t \leq t_2$, we shall refer to the family of elastostatic

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[1] Owing to the continuity of the displacements, the balance of mass on either side of $\mathcal{L}(t)$ assures the mass balance across $\mathcal{L}(t)$.

fields characterized by (6.3), (6.4), together with the continuity conditions (6.6), (6.7), as a time-dependent (piecewise homogeneous) elastostatic shock with the (moving) material and spatial shock-lines[1] $\mathcal{L}(t)$ and $\mathcal{L}_*(t)$ $(t_1 \leq t \leq t_2)$. Observe that at any fixed instant t such a shock is an equilibrium shock of the type defined in Section 3, if $\underset{\sim}{p}(t) = \underset{\sim}{0}$ and the translation vectors $\overset{+}{\underset{\sim}{b}}(t), \overset{-}{\underset{\sim}{b}}(t)$ also vanish.

It is readily seen that equations (3.8) through (3.15) hold identically on $[t_1, t_2]$ for time-dependent shocks provided $\underset{\sim}{\ell}, \underset{\sim}{n}, \Phi, \varphi$ and all field quantities retain their previous meaning but are now regarded as functions of time. On the other hand, at any fixed instant the mapping (6.3) admits the resolution

$$\left. \begin{array}{l} \underset{\sim}{z} = \overset{+}{\underset{\sim}{F}}(t)\underset{\sim}{x} + \overset{+}{\underset{\sim}{b}}(t) \text{ for all } \underset{\sim}{x} \text{ in } \mathcal{R}, \\[2em] \underset{\sim}{y} = \begin{cases} \underset{\sim}{z} \text{ for all } \underset{\sim}{z} \text{ in } \overset{+}{\mathcal{R}}_*(t) \\[1em] \underset{\sim}{B}(t)\underset{\sim}{z} - \underset{\sim}{B}(t)\overset{+}{\underset{\sim}{b}}(t) + \overset{-}{\underset{\sim}{b}}(t) \text{ for all } \underset{\sim}{z} \text{ in } \bar{\mathcal{R}}_*(t), \end{cases} \end{array} \right\} \quad (6.8)$$

where $\overset{+}{\mathcal{R}}_*(t)$ and $\bar{\mathcal{R}}_*(t)$ are the instantaneous deformation images of $\overset{+}{\mathcal{R}}(t)$ and $\bar{\mathcal{R}}(t)$ associated with (6.3), while $\underset{\sim}{B}$ obeys the first of (3.15) on $[t_1, t_2]$. Arguments strictly parallel to those employed in Section 3 confirm that equations (3.18) through (3.37) are valid on $[t_1, t_2]$ in the present circumstances. Note that the frame X' with the base vectors $(\underset{\sim}{\ell}, \underset{\sim}{n})$ is now time-dependent, as are the parameters $\varkappa, \delta,$ and ϵ, whose kinematic significance remains unaltered. Figure 6 is applicable to the instantaneous supplementary deformation inherent in the decomposition (6.8), except that the spatial shock-line $\mathcal{L}_*$, while parallel to — need not be coincident with the y_1'-axis.

Consider a time-dependent elastostatic shock of duration $[t_1, t_2]$ and

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[1] The term "material shock-line" is somewhat misleading in the present setting since $\mathcal{L}(t)$, being occupied by different particles at different times, is not a material line.

let t_o be in the open interval (t_1, t_2). With a view toward arriving at the notion of dissipativity of the given shock at time t_o we adopt the following definition. A domain $\mathfrak{D}$ of the (x_1, x_2)-plane is a <u>test</u> <u>domain</u> <u>admissible</u> <u>at</u> <u>time</u> t_o, if $\mathfrak{D}$ is the interior of a <u>fixed</u>, smooth and convex, simple closed curve C and the material shock-line $\mathcal{L}(t_o)$ intersects $\mathfrak{D}$. Then, evidently, $\mathcal{L}(t)$ intersects $\mathfrak{D}$ for all values of t in a neighborhood of t_o, which we shall call a "time-range of intersection of $\mathfrak{D}$." Further, for every t in such a range of time, $\mathcal{L}(t)$ intersects the convex boundary C of $\mathfrak{D}$ in two distinct points — say with the position vectors $\hat{\underset{\sim}{x}}(\xi_1(t), t)$ and $\hat{\underset{\sim}{x}}(\xi_2(t), t)$, where $\xi_1(t) < \xi_2(t)$, as indicated in Figure 9. Also, $\mathcal{L}(t)$ divides the <u>stationary</u> domain $\mathfrak{D}$ into the two <u>time-dependent</u> sub-domains $\overset{+}{\mathfrak{D}}(t)$ and $\bar{\mathfrak{D}}(t)$, which are the intersections of $\mathfrak{D}$ with the half-planes $\overset{+}{\mathfrak{R}}(t)$ and $\bar{\mathfrak{R}}(t)$, respectively (see Figure 9).

At this stage we choose a test domain $\mathfrak{D}$ admissible at the given instant and let $U(t)$ stand for the total strain energy in a slab of unit thickness — at any time t within a time-range of intersection of $\mathfrak{D}$ — stored in the material that occupies $\mathfrak{D}$ in the undeformed configuration. Accordingly, we have

$$U(t) = \overset{+}{W}(t)\overset{+}{A}(t) + \bar{W}(t)\bar{A}(t) \quad , \quad \overset{+}{W}(t) = W(\overset{+}{\underset{\sim}{F}}(t)) \quad , \quad \bar{W}(t) = W(\bar{\underset{\sim}{F}}(t)), \tag{6.9}$$

where $\overset{+}{A}(t)$ and $\bar{A}(t)$ are evidently the respective areas of $\overset{+}{\mathfrak{D}}(t)$ and $\bar{\mathfrak{D}}(t)$. Differentiation of the first of (6.9), in view of (6.4) and the second of (6.9), yields the time-rate of change of the strain energy U in the form

$$U^{\boldsymbol{\cdot}} = \overset{+}{W}\overset{+}{A}{}^{\boldsymbol{\cdot}} + \bar{W}\bar{A}^{\boldsymbol{\cdot}} + \overset{+}{\sigma}_{\beta\gamma}\overset{+}{F}{}^{\boldsymbol{\cdot}}_{\beta\gamma}\overset{+}{A} + \bar{\sigma}_{\beta\gamma}\bar{F}^{\boldsymbol{\cdot}}_{\beta\gamma}\bar{A}. \tag{6.10}[1]$$

From the divergence theorem follows at once

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[1] Here and in the sequel dots used as superscripts indicate differentiation with respect to the time.

$$\delta_{\alpha\beta} \overset{+}{A}(t) = \oint_{\partial \overset{+}{\mathcal{D}}(t)} x_\alpha \nu_\beta d\mathfrak{s} \quad , \quad \delta_{\alpha\beta} \overset{-}{A}(t) = \oint_{\partial \overset{-}{\mathcal{D}}(t)} x_\alpha \nu_\beta d\mathfrak{s},$$ (6.11)

if $\mathfrak{s}$ is the arc-length along the boundaries $\partial \overset{+}{\mathcal{D}}(t)$, $\partial \overset{-}{\mathcal{D}}(t)$ of the domains $\overset{+}{\mathcal{D}}(t)$, $\overset{-}{\mathcal{D}}(t)$ and $\underset{\sim}{\nu}$ is the appropriate unit outward normal vector. On the other hand, it is readily shown, and intuitively obvious, that

$$\overset{+}{A}{}^{\bullet}(t) = -\overset{-}{A}{}^{\bullet}(t) = -\int_{\xi_1(t)}^{\xi_2(t)} \underset{\sim}{V}(\xi, t) \cdot \underset{\sim}{N}(t) d\xi,$$ (6.12)

provided

$$\underset{\sim}{V}(\xi, t) = \frac{\partial}{\partial t} \underset{\sim}{\hat{x}}(\xi, t),$$ (6.13)

so that[1] $\underset{\sim}{V}(\xi, t)$ is the velocity, relative to the frame X, of the point on the moving shock-line $\mathcal{L}(t)$ with the position vector $\underset{\sim}{\hat{x}}(\xi, t)$.

We now use (6.11), (6.12) in conjunction with (6.10) and then invoke (6.2), the continuity conditions (6.5), (6.7), as well as the overall equilibrium of the material occupying $\overset{+}{\mathcal{D}}(t)$ and $\overset{-}{\mathcal{D}}(t)$. In this manner we are led to

$$U^{\bullet}(t) = \int_{\partial \mathcal{D}} \underset{\sim}{s}(\underset{\sim}{x}, t) \cdot \underset{\sim}{v}(\underset{\sim}{x}, t) d\mathfrak{s} + \int_{\xi_1(t)}^{\xi_2(t)} \{ [W(t)]_{+}^{-} \underset{\sim}{V}(\xi, t) \cdot \underset{\sim}{N}(t) + \underset{\sim}{s}(t) \cdot [\underset{\sim}{v}(\underset{\sim}{x}, t)]_{+}^{-} \} d\xi.$$ (6.14)

Here $\underset{\sim}{s}(\underset{\sim}{x}, t)$ is the Piola traction on the outer side of $\partial \mathcal{D}$, $\underset{\sim}{v}(\underset{\sim}{x}, t)$ is the particle velocity given by

$$\underset{\sim}{v}(\underset{\sim}{x}, t) = \frac{\partial}{\partial t} \underset{\sim}{\chi}(\underset{\sim}{x}, t) = \begin{cases} \overset{+}{\underset{\sim}{F}}{}^{\bullet}(t)\underset{\sim}{x} + \overset{+}{\underset{\sim}{b}}{}^{\bullet}(t) \text{ on } \overset{+}{\mathcal{R}}(t) \\ \\ \overset{-}{\underset{\sim}{F}}{}^{\bullet}(t)\underset{\sim}{x} + \overset{-}{\underset{\sim}{b}}{}^{\bullet}(t) \text{ on } \overset{-}{\mathcal{R}}(t), \end{cases}$$ (6.15)

while the notation $[\ ]_{+}^{-}$ is employed at present to denote the jump in the

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[1] Recall (6.2).

appropriate field values as $\mathcal{L}(t)$ is traversed from $\overset{+}{\mathcal{R}}(t)$ to $\overset{-}{\mathcal{R}}(t)$; thus, in particular,

$$[W(t)]_+^- = W(\overset{-}{\underset{\sim}{F}}(t)) - W(\overset{+}{\underset{\sim}{F}}(t)) \ , \ [\underset{\sim}{v}(\underset{\sim}{x}, t)]_+^- = [\underset{\sim}{F}^{\bullet}(t)]_+^- \overset{\hat{}}{\underset{\sim}{x}}(\xi, t) + [\underset{\sim}{b}^{\bullet}(t)]_+^- . \qquad (6.16)$$

From the second of (6.16), together with (6.2), (6.5), and (6.13) follows

$$[\underset{\sim}{v}(\underset{\sim}{x}, t)]_+^- = -[\underset{\sim}{F}(t)]_+^- \underset{\sim}{V}(\xi, t). \qquad (6.17)$$

Substituting from (6.17) into (6.14), and appealing once more to the traction-continuity condition (6.7), one finds after elementary manipulations:

$$U^{\bullet}(t) = \int_{\partial \mathcal{D}} \underset{\sim}{s}(\underset{\sim}{x}, t) \cdot \underset{\sim}{v}(\underset{\sim}{x}, t) d\mathscr{S} + \int_{\xi_1(t)}^{\xi_2(t)} [\underset{\sim}{P}(t)]_+^- \underset{\sim}{N}(t) \cdot \underset{\sim}{V}(\xi, t) d\xi, \qquad (6.18)$$

provided

$$\underset{\sim}{P}(\underset{\sim}{F}) = W(\underset{\sim}{F})\underset{\sim}{1} - \underset{\sim}{F}^T \underset{\sim}{\sigma}(\underset{\sim}{F}) \ , \ \overset{+}{\underset{\sim}{P}}(t) = \underset{\sim}{P}(\overset{+}{\underset{\sim}{F}}(t)) \ , \ \overset{-}{\underset{\sim}{P}}(t) = \underset{\sim}{P}(\overset{-}{\underset{\sim}{F}}(t)). \qquad (6.19)$$

The <u>first</u> integral in (6.18) is evidently the power of the tractions external with respect to the material occupying $\mathcal{D}$ in the reference configuration; the <u>second</u> integral, which vanishes when $\overset{+}{\underset{\sim}{F}}(t) = \overset{-}{\underset{\sim}{F}}(t)$, represents the contribution to $U^{\bullet}(t)$ arising from the deformation-gradient discontinuity at $\mathcal{L}(t)$. Moreover, $\underset{\sim}{P}$ is the energy-momentum tensor originally introduced by Eshelby [10][1] into the theory of defects in elastic bodies. Eshelby established the relation between the tensor $\underset{\sim}{P}$ and the "force exerted on the defect"; this relation — in the absence of such defects — gives rise to a familiar conservation law in finite and linearized elastostatics. The two-dimensional version of the conservation law to which we are alluding asserts

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[1] This recent expository paper contains references to Eshelby's related earlier work.

that

$$J \equiv \oint_\Gamma P(F)N \, d\phi = 0 \tag{6.20}$$

for all plane equilibrium deformations (of a homogeneous hyperelastic solid) with twice continuously differentiable displacements, if Γ is the boundary of an arbitrary regular region within the domain of regularity of the elastostatic field at hand and N is the outward unit normal vector of Γ. The fracture-mechanical implications of (6.20) and, in particular, the connection between the so-called J-integral and the energy release rate at the tip of a crack, were first recognized by Rice [11].

For our purposes it is useful to cast the energy identity (6.18) into a different form. To this end we use (6.19), (6.7), and the first of (6.6) to infer that

$$[P(t)]_+^- N(t) \cdot L(t) = -s(t) \cdot [F(t)]_+^- L(t) = 0, \tag{6.21}$$

whence

$$[P(t)]_+^- N(t) = \{[P(t)]_+^- N(t) \cdot N(t)\} N(t). \tag{6.22}$$

Accordingly, the energy identity (6.18) is equivalent to

$$U^\cdot(t) = \int_{\partial \Omega} s(x, t) \cdot v(x, t) \, d\phi - [H(t)]_+^- \int_{\xi_1(t)}^{\xi_2(t)} V(\xi, t) \cdot N(t) \, d\xi, \tag{6.23}$$

where

$$\left.\begin{array}{l} H(F, N) = -P(F)N \cdot N = -W(F) + FN \cdot \sigma(F)N, \\[2mm] \overset{+}{H}(t) = H(\overset{+}{F}(t), N(t)) \,, \quad \bar{H}(t) = H(\bar{F}(t), N(t)). \end{array}\right\} \tag{6.24}$$

In view of (6.24) and (6.7) one has

$$[H(t)]_+^- = s(t) \cdot [F(t)]_+^- N(t) - [W(t)]_+^-, \tag{6.25}$$

so that the jump in H(t) is the excess, over the jump in the strain-energy density W(t), of the "internal work" done by the Piola traction $\underset{\sim}{s}(t)$ through the jump $[\underset{\sim}{F}(t)]_+^- \underset{\sim}{N}(t)$ in the deformation gradient normal to the material shock-line $\mathcal{L}(t)$.

Bearing in mind the physical significance of the second term in (6.23) as the contribution to the energy rate $U^\bullet(t)$ due to the deformation-gradient discontinuity at the shock-line $\mathcal{L}(t)$, we introduce the following definition. A time-dependent elastostatic shock is dissipative at an instant t in the interior of its interval of duration $[t_1, t_2]$ if and only if

$$[H(t)]_+^- \int_{\xi_1(t)}^{\xi_2(t)} \underset{\sim}{V}(\xi, t) \cdot \underset{\sim}{N}(t) d\xi > 0 \tag{6.26}$$

for every test domain $\mathcal{D}$ admissible at this instant.

We now establish necessary and sufficient conditions in order that a time-dependent piecewise homogeneous elastostatic shock be dissipative at a given instant. Suppose such a shock is dissipative at t. Then, we show first, its material shock-line $\mathcal{L}$ must be in a state of translation at that moment and the direction of this instantaneous translation cannot be parallel to $\mathcal{L}(t)$:

$$\underset{\sim}{V}(\xi, t) = \underset{\sim}{V}(0, t) \equiv \overset{\bullet}{\underset{\sim}{V}}(t) \ \ (-\infty < \xi < \infty) \ , \ \ \overset{\bullet}{\underset{\sim}{V}}(t) \cdot \underset{\sim}{N}(t) \neq 0 . \tag{6.27}$$

Indeed, if the first of (6.27) were false, there would exist an instantaneous center of rotation for the rigid motion of $\mathcal{L}$ at the time t. In this event $\underset{\sim}{V}(\xi, t) \cdot \underset{\sim}{N}(t)$ would range continuously over all real numbers as ξ ranges over $(-\infty, \infty)$ and thus the inequality (6.26) could not possibly hold for every $\mathcal{D}$ that is the interior of a sufficiently small circle centered on $\mathcal{L}(t)$. Further,

the first of (6.27) is incompatible with (6.26) unless the second of (6.27) holds true. Next, from (6.26), (6.27) follows

$$[H(t)]_+^- > 0 \qquad (6.28)$$

provided $\mathcal{L}(t)$ is oriented by means of $\underset{\sim}{L}(t)$ in such a way that

$$\overset{\circ}{\underset{\sim}{V}}(t) \cdot \underset{\sim}{N}(t) > 0. \qquad (6.29)$$

Consequently, $H(t)$ must increase as $\mathcal{L}(t)$ is traversed from $\overset{+}{\mathcal{R}}(t)$ to $\overset{-}{\mathcal{R}}(t)$, if $\overset{+}{\mathcal{R}}(t)$ is understood to be the half-plane into which $\mathcal{L}$ is advancing at the instant under consideration. Also, (6.28), (6.29) and the first of (6.27) are evidently <u>sufficient</u> for the dissipativity of the shock at time t. Finally, it is clear from the above that a piecewise homogeneous time-dependent shock cannot be dissipative throughout a time-interval (t_1, t_2) unless the motion of its material shock-line is purely translatory.

The dissipation inequality (6.28), as will become apparent later on, assumes a role analogous to that played by the entropy inequality in connection with gas-dynamical shocks. Our immediate goal is to deduce a useful alternative representation for the jump $[H(t)]_+^-$ appropriate to a time-dependent elastostatic shock. The subsequent considerations apply to any fixed instant in the time interval (t_1, t_2); for the sake of brevity we shall suppress the argument t in the equations to follow. Because of (6.24) and (3.3), (3.32) we are entitled to write

$$H(\underset{\sim}{F}, \underset{\sim}{N}) = -W(\underset{\sim}{F}) + J(\underset{\sim}{F})\underset{\sim}{F}\underset{\sim}{N} \cdot \underset{\sim}{\tau}(\underset{\sim}{F})(\underset{\sim}{F}^{-1})^T\underset{\sim}{N} \text{ if } \underset{\sim}{F} = \overset{+}{\underset{\sim}{F}} \text{ or } \underset{\sim}{F} = \overset{-}{\underset{\sim}{F}}, \qquad (6.30)$$

where $\underset{\sim}{\tau}(\underset{\sim}{F})$ is the actual stress field associated with the nominal stress field $\underset{\sim}{\sigma}(\underset{\sim}{F})$. On the other hand, (3.11), (3.12), (3.13) and the traction continuity condition in the form (3.33) easily lead from (6.30) to

$$H(\underset{\sim}{F}, \underset{\sim}{N}) = -W(\underset{\sim}{F}) + \frac{J(\underset{\sim}{F}) \underset{\sim}{t}(\underset{\sim}{F}) \cdot \underset{\sim}{G}(\underset{\sim}{F}) \underset{\sim}{n}}{\underset{\sim}{n} \cdot \underset{\sim}{G}(\underset{\sim}{F}) \underset{\sim}{n}} \quad \text{if } \underset{\sim}{F} = \overset{+}{\underset{\sim}{F}} \text{ or } \underset{\sim}{F} = \bar{\underset{\sim}{F}}, \tag{6.31}$$

in which $\underset{\sim}{n}$ is the unit normal vector of the (moving) spatial shock-line $\mathcal{L}_*$ and $\underset{\sim}{t}(\overset{+}{\underset{\sim}{F}}) = \underset{\sim}{t}(\bar{\underset{\sim}{F}})$ the actual traction vector along $\mathcal{L}_*$. Recall now that $\underset{\sim}{\ell}$ is the unit direction vector of $\mathcal{L}_*$ and X' the (time-dependent) coordinate frame with the base vectors $(\underset{\sim}{\ell}, \underset{\sim}{n})$ and the same origin as X. On expanding the two scalar products entering (6.31) in X', noting that $\underset{\sim}{n}$ and $\underset{\sim}{t}$ have the components $(0, 1)$ and (τ'_{12}, τ'_{22}) in this frame, we obtain

$$H(\underset{\sim}{F}, \underset{\sim}{N}) = -W(\underset{\sim}{F}) + J(\underset{\sim}{F}) \left\{ \frac{G'_{12}(\underset{\sim}{F})}{G'_{22}(\underset{\sim}{F})} \tau'_{12}(\underset{\sim}{F}) + \tau'_{22}(\underset{\sim}{F}) \right\} \quad \text{if } \underset{\sim}{F} = \overset{+}{\underset{\sim}{F}} \text{ or } \underset{\sim}{F} = \bar{\underset{\sim}{F}}. \tag{6.32}$$

Finally, from (6.32) and (3.25), (3.27), the continuity across $\mathcal{L}_*$ of the stress components τ'_{12}, τ'_{22}, and the last two of (6.24), follows the useful result

$$[H]^-_+ = -[W]^-_+ + \overset{+}{J}(\varkappa \tau'_{12} - \varepsilon \tau'_{22}), \quad \varepsilon = 1 - \delta, \tag{6.33}$$

where

$$\overset{+}{\tau}'_{12} = \tau'_{12}(\overset{+}{\underset{\sim}{F}}), \quad \overset{+}{\tau}'_{22} = \tau'_{22}(\overset{+}{\underset{\sim}{F}}). \tag{6.34}$$

In (6.33), $\varkappa$ and δ are the values of the amount of shear and of the relative stretch at right angles to $\mathcal{L}_*$ inherent in the instantaneous supplementary deformation.[1]

At this point we employ (6.33) in applying the dissipation condition (6.28) to the special case of a piecewise homogeneous time-dependent elastostatic shock occurring in <u>Material</u> 1. To this end we observe first that the results of Section 5 continue to hold at present if proper allowance is made for the time-dependence of the equilibrium shock. In particular, the

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[1] Refer to the decomposition (6.8) and to (3.19), (3.28). See also Figure 6.

jump $[W(t)]^-_+$ in the energy density is available from (5.22) if in this equation $\lambda_1, \lambda_2, \beta, \varphi$, and δ are replaced by $\lambda_1(t), \lambda_2(t), \beta(t), \varphi(t)$, and $\delta(t)$. Further, (1.14), (1.18) and (4.41) yield the stress components

$$\overset{+}{\tau}'_{12} = \frac{\mu}{\overset{+}{J}3}\overset{+}{G}'_{12} , \quad \overset{+}{\tau}'_{22} = \mu \left\{1 - \frac{\overset{+}{G}'_{11}}{\overset{+}{J}3}\right\} \quad \text{on } [t_1, t_2]. \tag{6.35}$$

The required jump in H(t) may now be calculated from (6.33) with the aid of (5.22), (6.35), the formulas (5.4) for $\overset{+}{G}'_{\alpha\beta}$, and equation (5.8) for the amount of supplementary shear $\varkappa$. This computation ultimately gives

$$[H]^-_+ = \frac{\mu(1-\beta^2)(2-\epsilon)\epsilon^3}{2\overset{+}{J}(1-\epsilon)^3(1+2\beta\cos2\varphi+\beta^2)} \quad \text{on } [t_1, t_2] , \quad \epsilon = 1-\delta. \tag{6.36}$$

Now $\mu>0, \overset{+}{J}>0$, whereas $|\beta|<1$ and $\epsilon<1$ according to (5.6). Hence (6.36) implies

$$[H(t)]^-_+ > 0 \text{ if and only if } \epsilon(t) > 0, \tag{6.37}$$

so that the dissipation condition (6.28), which rests on the assumption that the shock is advancing into $\overset{+}{\Re}(t)$ at the time t, leads to the shock-strength restriction $\epsilon(t)>0$ in the case of Material 1 (for all admissible homogeneous deformations on $\overset{+}{\Re}(t)$). This restriction was anticipated in Section 5, when the detailed discussion of some of the results deduced there was confined to non-negative values of ϵ.

One gathers from (6.36) that for Material 1,

$$[H]^-_+ = O(\epsilon^3) \text{ as } \epsilon \to 0 \tag{6.38}$$

at any particular instant. The general validity of this estimate, even for elastostatic shocks in _anisotropic_ hyperelastic materials, can be established

by means of an appropriate weak-shock expansion[1]. Furthermore, (6.38) is the analogue of a familiar property of the entropy jump in gas-dynamical shocks.

We now list various implications of the conclusion that $\epsilon(t)>0$ for a dissipative shock in Material 1 if the shock happens to be advancing into $\overset{+}{\mathfrak{R}}(t)$ at this moment. The following results apply to the instant under consideration, although the argument t will be suppressed.

First, in view of the kinematic significance of the shock-strength ϵ, the supplementary deformation involves a contraction at right angles to the spatial shock-line $\mathfrak{L}_{*}$. Second, it is clear from (5.3) and (5.6), (5.8) that the sign of the supplementary amount of shear $\varkappa$ depends on the nature of the instantaneous principal stretches inherent in the given homogeneous deformation on $\overset{+}{\mathfrak{R}}$. One has

$$\varkappa<0 \text{ if } \beta>0 \ (\lambda_2>\lambda_1) \ , \ \varkappa>0 \text{ if } \beta<0 \ (\lambda_2<\lambda_1) \tag{6.39}$$

for shocks with a spatial shock-angle in the first quadrant $(0, \pi/2)$.

Next, from $(4.4)^2$, (5.6), and (5.22) one draws

$$[J]_+^-<0 \ , \ [\rho]_+^->0 \ , \ [W]_+^-<0. \tag{6.40}$$

Consequently, in the present instance, the area-ratio and the energy density decrease, while the mass density increases — as the material shock-line $\mathfrak{L}$ is traversed from $\overset{+}{\mathfrak{R}}$ to $\overset{-}{\mathfrak{R}}$. Further, we recall from the discussion at the end of Section 5 that in the case of Material 1, $\epsilon>0$ assures a loss of ellipticity on $\overset{+}{\mathfrak{R}}$ at all admissible deformations on $\overset{+}{\mathfrak{R}}$; in contrast,

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[1] For this purpose one requires a weak-shock expansion of higher order than that considered in Section 4.

[2] Note that (4.4) holds true for shocks of finite strength.

the ensuing deformation on $\bar{\mathcal{R}}$ may or may not be accompanied by a failure of ellipticity of the displacement equations of equilibrium. Thus, at least for Material 1, the situation is parallel to that encountered in gas-dynamical shocks, where — as a consequence of the entropy inequality — the flow is necessarily supersonic on the upstream side but may be either supersonic or subsonic on the downstream side.

Finally, consider a time-dependent piecewise homogeneous equilibrium shock in Material 1 that is dissipative at <u>all</u> times in the interior of its interval of duration $[t_1, t_2]$. Suppose, in addition, that the material shock-line $\mathcal{L}(t)$ of the shock advances steadily into $\overset{+}{\mathcal{R}}(t)$ for $t_1 < t < t_2$. Then, $\epsilon > 0$ on (t_1, t_2) and — as was shown earlier in this section — the motion of $\mathcal{L}$ is one of pure translation throughout this range of time. Thus, the material shock-angle obeys

$$\Phi(t) = \Phi_o \quad (t_1 < t < t_2), \tag{6.41}$$

in which Φ_o is a constant. In these circumstances according to (3.14), for a <u>given</u> (admissible) deformation on $\overset{+}{\mathcal{R}}(t)$ $(t_1 \le t \le t_2)$, the spatial shock-angle φ becomes a known function of Φ_o and the time, i.e., $\varphi = \varphi(t; \Phi_o)$ $(t_1 < t < t_2)$. Also, one confirms easily that the first of (5.9) may now be inverted to give $\epsilon = \epsilon(t; \Phi_o)$, so that the shock-strength becomes a fully determinate function of time for every fixed value of Φ_o. We shall not explore the specific properties of $\epsilon(t; \Phi_o)$, however, since the foregoing considerations are strictly limited to <u>piecewise</u> <u>homogeneous</u> shocks and have no local analogue in connection with more general equilibrium shocks.

For comparison purposes we cite also, without proof, some results obtained by applying the dissipation condition (6.28) to a <u>weak</u>

time-dependent shock in Material 2. Here one finds that $\epsilon < 0$, so that the instantaneous supplementary deformation involves a _stretching_ perpendicular to $\mathcal{L}_*$. On the other hand, (4.42) and (4.36) imply that the inequalities (6.39), governing the sign of the supplementary amount of shear $\varkappa$, remain valid for Material 2 if $|\epsilon|$ is sufficiently small. Also, since $\epsilon < 0$, the first two inequalities in (6.40) now give way to

$$[J]_+^- > 0 \; , \; [\rho]_+^- < 0. \tag{6.42}$$

The sign of the energy-jump $[W]_+^-$ can no longer be inferred from the sign of ϵ alone in this special case; it is found to depend on the instantaneous character of the deformation pre-assigned on $\overset{+}{\mathcal{R}}$.

7. Equilibrium-shock formation as a bifurcation process. Lüders bands. Discussion.

The analysis in Section 4 led to the conclusion — valid under certain assumptions spelled out there — that the existence of a piecewise homogeneous elastostatic shock, and hence the emergence of discontinuities in the first deformation gradients, is contingent upon a breakdown of ellipticity in the displacement equations of equilibrium associated with the homogeneous deformation prescribed on $\overset{+}{\mathcal{R}}$. As is well known and easily verifiable, this failure of ellipticity in turn renders the given deformation dynamically unstable: there are initially periodic, small-amplitude, disturbances of the above uniform elastostatic field that give rise to solutions of the appropriate linearized displacement equations of motion which grow beyond bounds with time.

The foregoing state of affairs suggests the possibility of viewing the process of shock formation as a bifurcation from a homogeneous

equilibrium field in an elastic body. Such an interpretation of equilibrium shocks leads one to wonder about their relevance to the familiar failure phenomenon of Lüders bands, commonly observed in mild steels, which has received repeated analytical attention within plasticity theory. [1]

Lüders bands are known to develop[2] in specimens subjected to uni-axial tension or compression once the loads have exceeded the yield limit; their inclination relative to the load axis is often close to 45 degrees, al-though considerable departures from this angle have been reported. Fig-ure 10, which follows Nadai [13], shows a schematic diagram of Lüders bands. The experimental findings indicate abrupt changes in the deforma-tion gradients across the interfaces between such a band and the adjacent material, the deformation within each band being predominantly one of shear parallel to the interfaces. Further, the sense of this shear deforma-tion undergoes a reversal as the loading is changed from tension to com-pression.

In assessing the extent to which predictions based on the theory of piecewise homogeneous equilibrium shocks resemble experimental observa-tions pertaining to the formation of Lüders bands, it should be kept in mind that the actual test situation, which is in fact three-dimensional, comes closer to conditions of plane stress than to plane strain. Moreover, elementary equilibrium considerations preclude the existence of a homo-geneous field of deformation and stress in a Lüders band of finite width (see Figure 10) since such a uniform field is incompatible with the boundary

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[1]See, for example, Thomas [12].

[2]See Nadai [13], Chapter 18, for a detailed description of the pertinent experimental observations and for references to the previous phenomeno-logical literature on Lüders bands.

conditions at the load-free parallel edges of the specimen. For this reason a piecewise homogeneous elastostatic shock cannot possibly describe the observed behavior near the edges of the slab. On the other hand, it is a simple matter to symmetrize the elastostatic field of an equilibrium shock and thus arrive at a piecewise homogeneous field involving three homogeneous zones: two half-planes in which the same given deformation is sustained, separated by a strip that undergoes a distinct deformation, the displacements and tractions being continuous across the interfaces.

Despite the limitations pointed out above, the present theory of equilibrium shocks exhibits certain striking features that support its relevance (within the framework of continuum mechanics) as far as Lüders bands are concerned. The following considerations pertain to the local behavior near a point such as P in Figure 10, on an interface of a Lüders band.

To fix ideas and solely for illustrative purposes we draw once more on the special isotropic hyperelastic Material 1. Suppose an all-around infinite slab of this material occupies the entire (x_1, x_2)-plane $\mathfrak{R}$ in its undeformed configuration and is subjected to the time-dependent quasi-static pure homogeneous plane deformation:

$$\underset{\sim}{\chi} = \underset{\sim}{\chi}(\underset{\sim}{x}, t) = \underset{\sim}{A}(t)\underset{\sim}{x} \quad \text{for all } \underset{\sim}{x} \text{ in } \mathfrak{R} \quad (0 \le t < \infty), \tag{7.1}$$

where

$$\underset{\sim}{A}(t) = \begin{bmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{bmatrix} \quad (0 \le t < \infty), \tag{7.2}$$

$\underset{\sim}{A}(t)$ being the deformation-gradient tensor and $\lambda_\alpha(t)$ the principal stretches at time t. We shall assume that the foregoing deformation process starts

from the undeformed state at the instant $t=0$ and at all times thereafter corresponds to a state of uni-axial tension or compression parallel to the x_2-axis. Thus,[1]

$$\lambda_\alpha(0)=1 \ , \ \tau_{11}(t)=\sigma_{11}(t)=0 \ , \ \lambda_1(t)=\lambda_2^{-\frac{1}{3}}(t),$$

$$\tau_{22}(t)=\mu[1-\lambda_2^{-8/3}(t)] \ , \ \sigma_{22}(t)=\mu[\lambda_2^{-\frac{1}{3}}(t)-\lambda_2^{-3}(t)] \ \ (0\leq t<\infty). \tag{7.3}$$

In addition we require that in the case of uni-axial <u>tension</u>,

$$\lambda_2(t) \text{ is steadily increasing, } \lambda_2(t)\to\infty \text{ as } t\to\infty, \tag{7.4}$$

while in the case uni-axial <u>compression</u>

$$\lambda_2(t) \text{ is steadily decreasing, } \lambda_2(t)\to 0 \text{ as } t\to\infty. \tag{7.5}$$

We now recall from Section 2 (see Figure 1) that the homogeneous time-dependent equilibrium deformation characterized by (7.1), (7.2), (7.3) entails a loss of ellipticity whenever the stretch $\lambda_2(t)$ equals or exceeds a certain critical value in the tension case or fails to be above another critical value in the compression case. Hence, by virtue of (7.4), (7.5), in either case there is an instant t_o such that the deformation process under consideration is accompanied by a loss of ellipticity — and thus also by a loss of dynamic stability — at all times $t \geq t_o$. Moreover, it is now clear from the conclusions reached in Sections 5, 6 that if $t_1 \geq t_o$ the following bifurcation of the homogeneous time-dependent deformation at hand into a piecewise homogeneous time-dependent equilibrium shock becomes possible:

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[1] Recall (2.8), which apply to uni-axial stress parallel to the x_1-axis.

$$\chi = \chi(\underset{\sim}{x},t) = \begin{cases} \overset{+}{\underset{\sim}{F}}(t)\underset{\sim}{x} \text{ for all } \underset{\sim}{x} \text{ in } \overset{+}{\underset{\sim}{R}}(t), \ \overset{+}{\underset{\sim}{F}}(t) = \underset{\sim}{A}(t) \ (t_1 \le t < \infty) \\ \\ \bar{\underset{\sim}{F}}(t)\underset{\sim}{x} + \bar{\underset{\sim}{b}}(t) \text{ for all } \underset{\sim}{x} \text{ in } \bar{\underset{\sim}{R}}(t) \ (t_1 \le t < \infty), \end{cases}$$ (7.6)

provided

$$\overset{\circ}{\underset{\sim}{V}}(t) \cdot \underset{\sim}{N}(t) > 0 \quad (t_1 \le t < \infty).$$ (7.7)

Here $\overset{\circ}{\underset{\sim}{V}}(t)$ is the velocity of the material shock-line $\mathcal{L}(t)$ dividing R into the two half-planes $\overset{+}{\underset{\sim}{R}}(t)$, $\bar{\underset{\sim}{R}}(t)$; also $\mathcal{L}(t)$, whose unit normal vector $\underset{\sim}{N}(t)$ points into $\overset{+}{\underset{\sim}{R}}(t)$, is in translation because of the dissipativity of the shock. The assumption (7.7) amounts to the physically motivated requirement that the deformation in $\bar{\underset{\sim}{R}}(t)$ — associated with the local deformation in the Lüders band — steadily encroaches upon the as yet unencumbered deformation prevailing on $\overset{+}{\underset{\sim}{R}}(t)$. Because of (7.7) we may appeal to (6.37) and the dissipativity of the shock to infer that its strength $\varepsilon(t) > 0$ for $t_1 < t < \infty$.

The results pertaining to Material 1 in Sections 2, 4, 6 now permit various inferences that bear on the significance of equilibrium shocks in connection with Lüders bands. First, in the tension case the shock-bifurcation (7.6) can arise only after the nominal stress σ_{22} associated with the homogeneous deformation (7.1) has passed its peak (see Figure 1). Second, in both the tension and the compression case, the angle of inclination of the "emerging" spatial shock-line relative to the load-axis, i.e. the limiting value of this angle at zero shock-strength and at an incipient failure of ellipticity, is 45 degrees for Material 1. In general this angle is a material property and may be different for tension and compression.

Next, the shock-formation in the present instance involves a supplementary contraction at right angles to the interface between the half-planes separated by the spatial shock-line. On the other hand, according

to (6.39), the amount of shear $\varkappa$ accompanying each instantaneous supplementary deformation is positive[1] in the case of a compression-induced shock and negative for a tension-induced shock in Material 1. The last-mentioned conclusion is at variance with the usually observed deformation pattern of Lüders bands, schematically depicted in Figure 10. We do not know whether the opposite sense of over-all shear has been encountered in tests of actual materials. Be this as it may, one can show by asymptotic means that there are strain-energy densities, even within the limited class characterized by (2.1), that lead to $\varkappa > 0$ in weak tensile-shocks. The above predictions concerning the sign of $\varkappa$ therefore merely reflect a peculiarity of the particular idealized material under discussion[2].

The preceding remarks suggest a word of caution regarding the highly special nature of the two hypothetical materials used in this paper to illustrate the theory of elastostatic shocks. With a view toward physically realistic applications of the theory it would seem essential to explore a wider range of hyperelastic solids that can sustain a loss of ellipticity. In particular, it would be of interest to construct ideal materials that admit a loss of ellipticity in uni-axial tension and compression at pre-assignable stretch levels. Unfortunately, such an adaptability of the assumed constitutive behavior is bound to incur mounting analytical complexities.

The analysis carried out in this investigation may be generalized

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[1] See Figure 6(b) for the geometric meaning of $\varkappa > 0$.

[2] A weak tensile shock in Material 2 also involves $\varkappa < 0$, as follows from the observations at the end of Section 6. We recall that a compressive shock cannot occur in Material 2 since the latter does not suffer a loss of ellipticity in uni-axial compression.

to encompass no longer piecewise homogeneous plane elastostatic deformations with continuous displacement fields and merely piecewise continuous first and second displacement gradients, whose discontinuities are permitted to occur along curved shock-lines. Further, it is a priori clear that the results deduced here for piecewise homogeneous equilibrium shocks at once apply to the local situation at an interior point of a curved shock-line in a non-homogeneous elastostatic field. The emergence of weak shocks of this more general type evidently necessitates a breakdown of ellipticity in the displacement equations of equilibrium at the shock-line, which — in the weak-shock limit — must be a characteristic line associated with these equations at the prevailing deformation. Also, the dissipation inequality (6.28) remains locally valid in the present circumstances, but the dissipativity of such a shock no longer requires that the motion of the shock-line be translatory.

Presumably non-homogeneous equilibrium shocks would arise in boundary-value problems of finite elastostatics at loads that cause sufficiently severe local deformations — severe enough to induce a local failure of ellipticity in the governing field equations, provided of course the underlying elastic potential admits such a failure[1]. In the presence of geometric or material sources of stress concentrations (such as holes, notches, or inclusions) elastostatic shocks could thus evolve at comparatively moderate loads.

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[1] The situation envisaged here is analogous to that encountered in boundary-value problems for stationary flows in gas dynamics that give rise to super-sonic regions and concomitant shocks.

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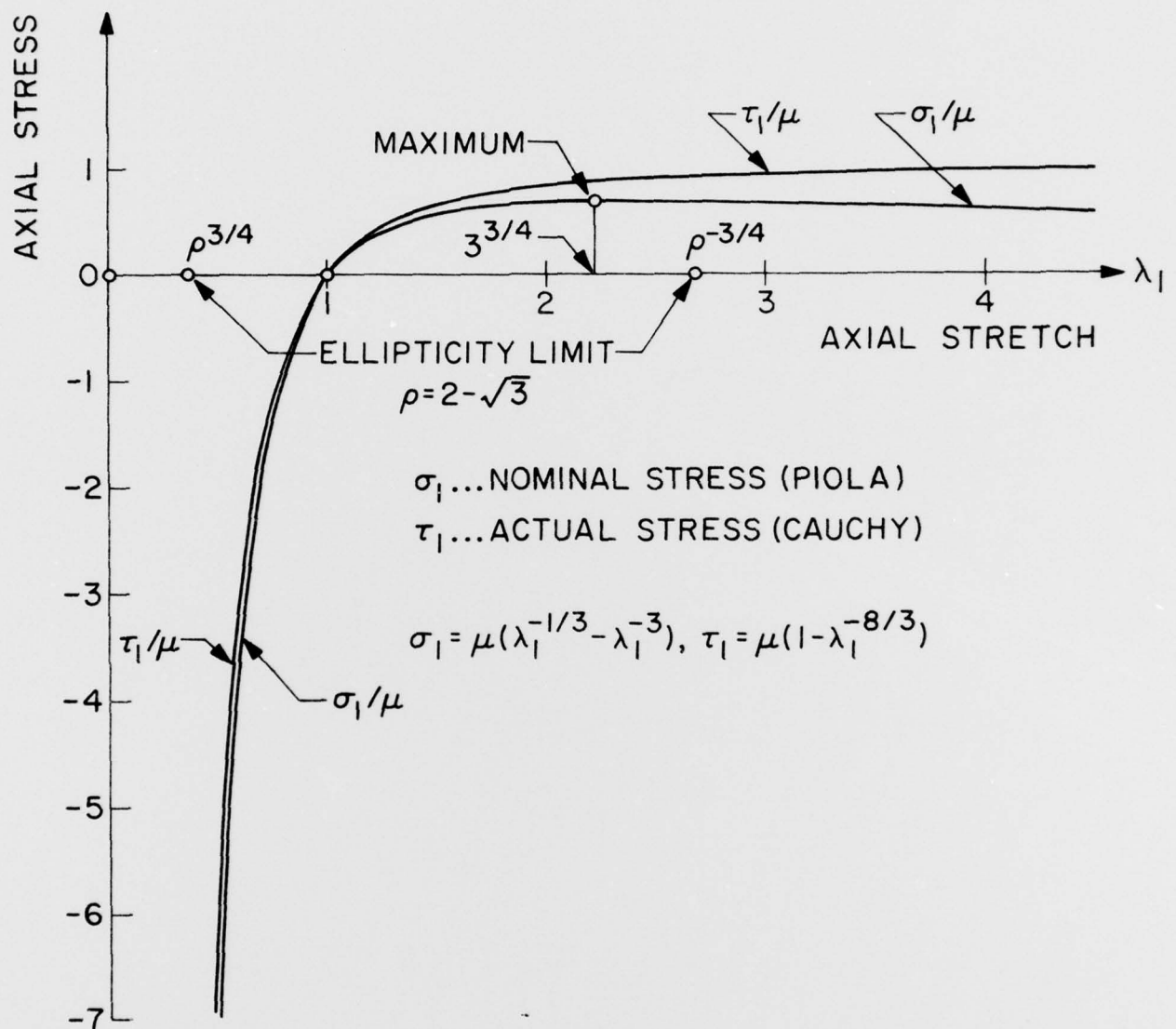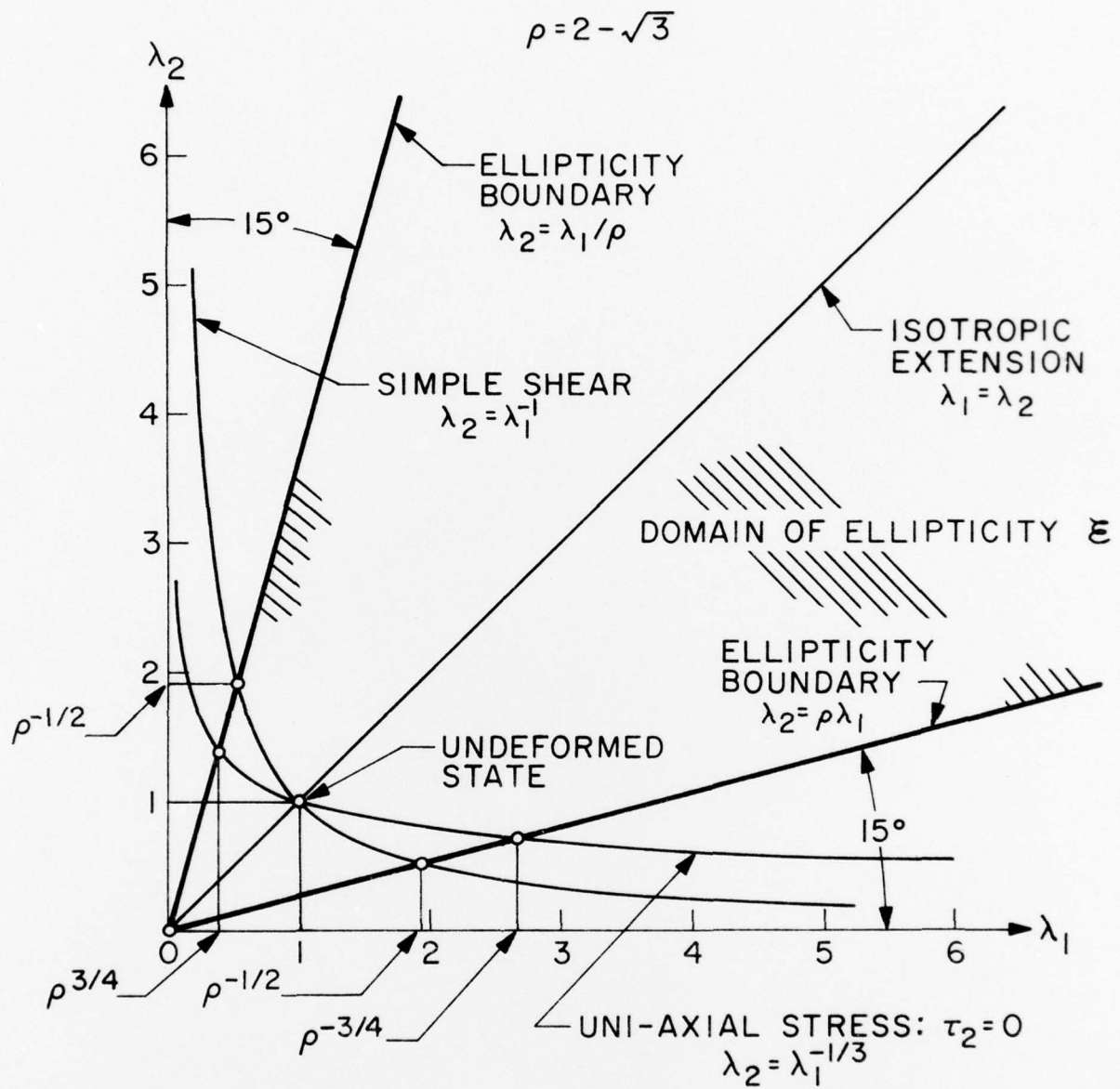[13] A. Nadai, Theory of flow and fracture of solids, Vol. 1, Second Edition, McGraw-Hill, New York, 1950.

AXIAL STRESS

MAXIMUM

τ_1/μ σ_1/μ

$\rho^{3/4}$ $3^{3/4}$ $\rho^{-3/4}$ λ_1

AXIAL STRETCH

ELLIPTICITY LIMIT
$\rho = 2 - \sqrt{3}$

$\sigma_1 \ldots$ NOMINAL STRESS (PIOLA)

$\tau_1 \ldots$ ACTUAL STRESS (CAUCHY)

τ_1/μ

σ_1/μ

$$\sigma_1 = \mu(\lambda_1^{-1/3} - \lambda_1^{-3}), \quad \tau_1 = \mu(1 - \lambda_1^{-8/3})$$

FIGURE 1. MATERIAL 1. PLANE-STRAIN UNI-AXIAL STRESS: $\tau_2 = 0$.
NOMINAL AND ACTUAL STRESS VS AXIAL STRETCH

$$\rho = 2 - \sqrt{3}$$

ELLIPTICITY BOUNDARY $\lambda_2 = \lambda_1/\rho$

$15°$

SIMPLE SHEAR $\lambda_2 = \lambda_1^{-1}$

ISOTROPIC EXTENSION $\lambda_1 = \lambda_2$

DOMAIN OF ELLIPTICITY $\mathcal{E}$

ELLIPTICITY BOUNDARY $\lambda_2 = \rho\lambda_1$

$15°$

UNDEFORMED STATE

$\rho^{-1/2}$

$\rho^{3/4}$ $\rho^{-1/2}$

$\rho^{-3/4}$

UNI-AXIAL STRESS: $\tau_2 = 0$ $\lambda_2 = \lambda_1^{-1/3}$

$$W(\lambda_1, \lambda_2) = \frac{\mu}{2}\left[2\lambda_1\lambda_2 + \lambda_1^{-2} + \lambda_2^{-2} - 4\right], \quad \overset{\circ}{\mu} = \mu, \ \overset{\circ}{\nu} = 1/4$$

FIGURE 2. MATERIAL I. PLANE-STRAIN DOMAIN OF ELLIPTICITY AND SPECIAL DEFORMATION PATHS
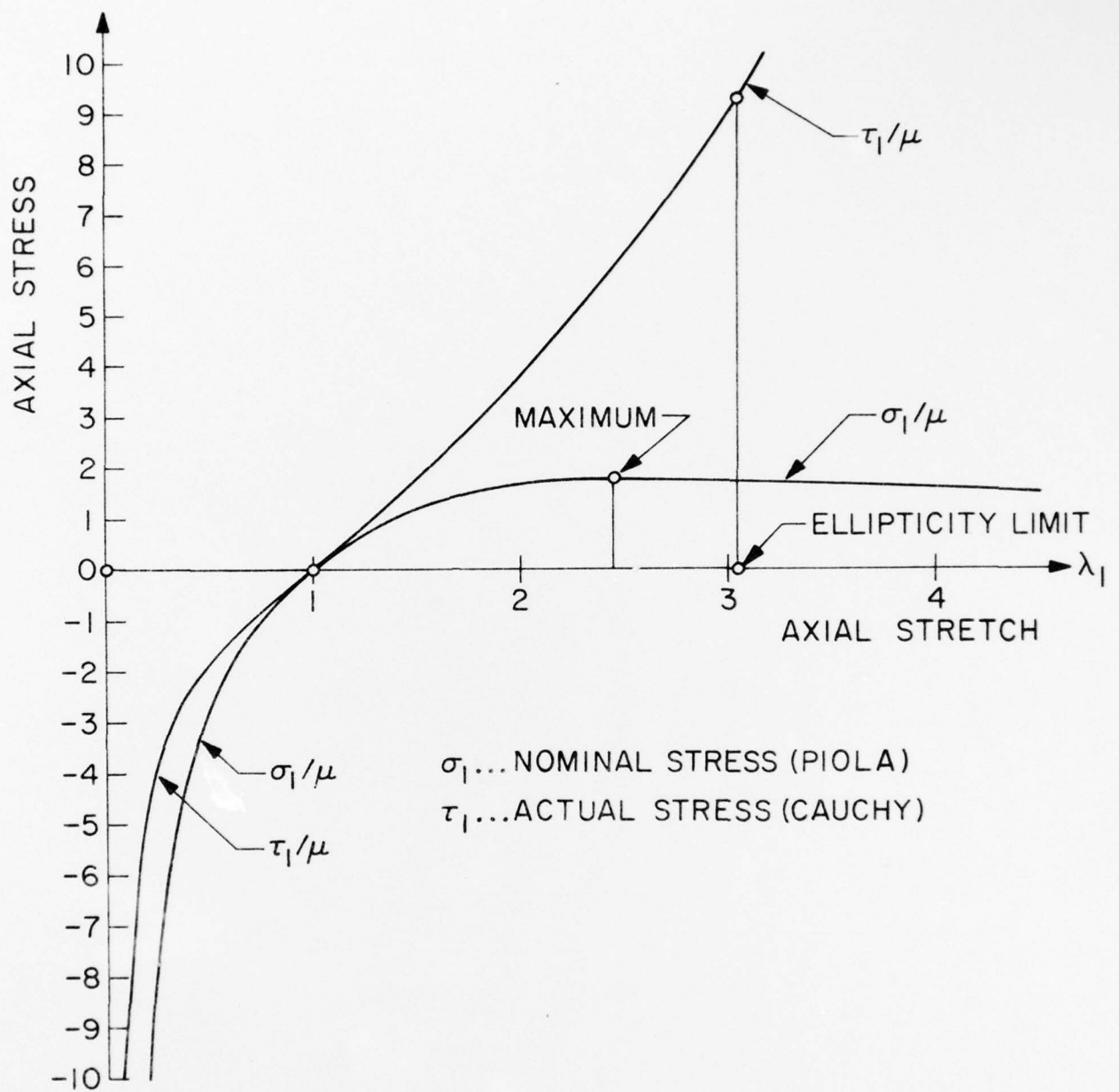
FIGURE 3. MATERIAL 2. PLANE-STRAIN UNI-AXIAL STRESS: $\tau_2 = 0$.
NOMINAL AND ACTUAL STRESS VS AXIAL STRETCH
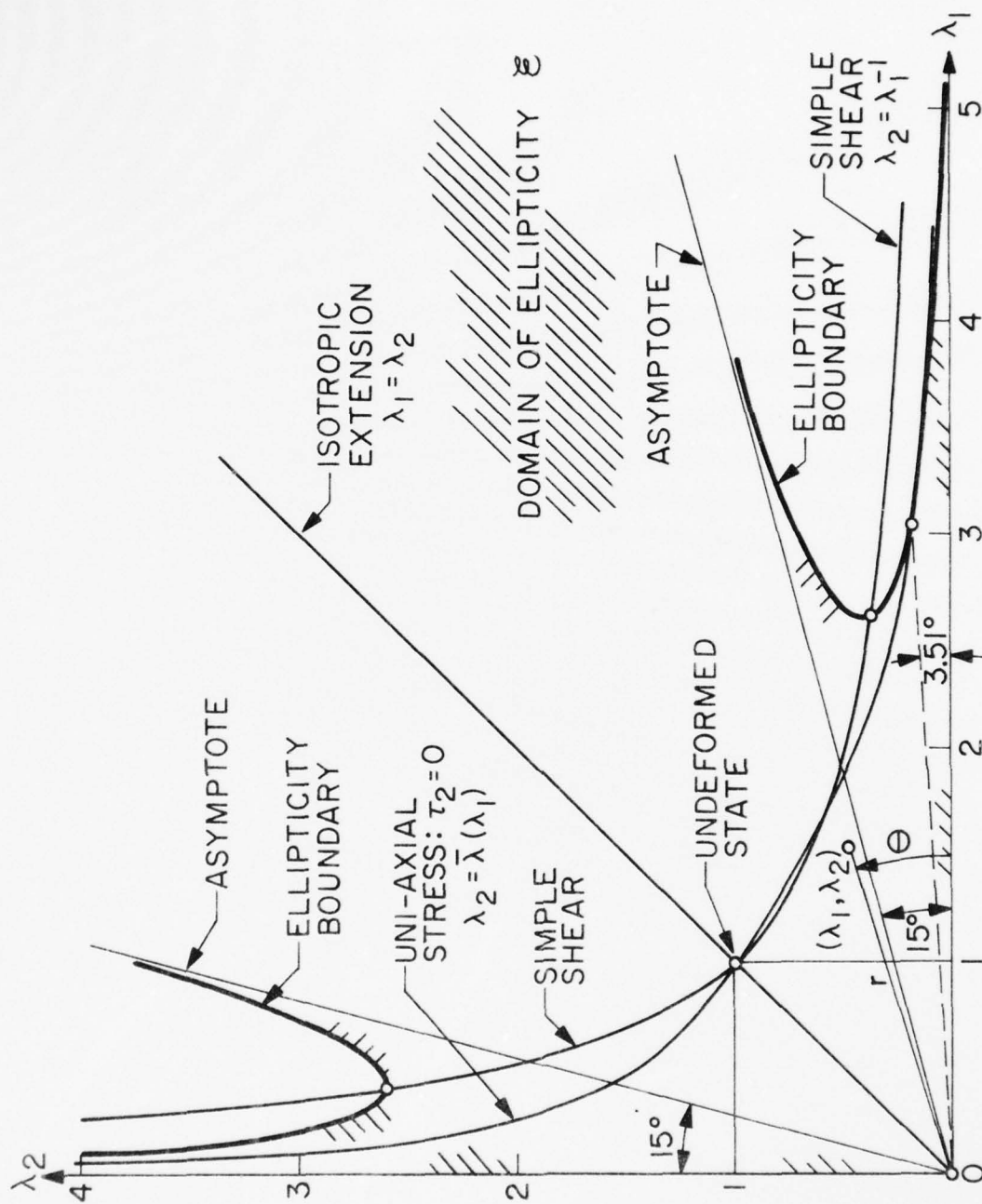
$$W(\lambda_1, \lambda_2) = \frac{\mu}{2} \left[\lambda_1^3 \lambda_2 + \lambda_1 \lambda_2^3 + 8(\lambda_1 \lambda_2)^{-1/2} - 10 \right], \quad \overset{\circ}{\mu} = \mu, \quad \overset{\circ}{\nu} = 2/5$$

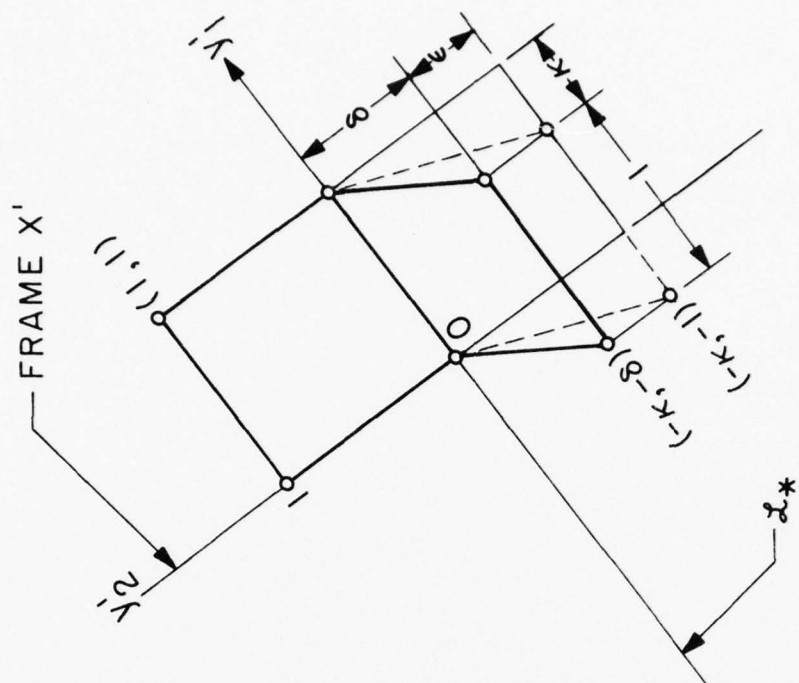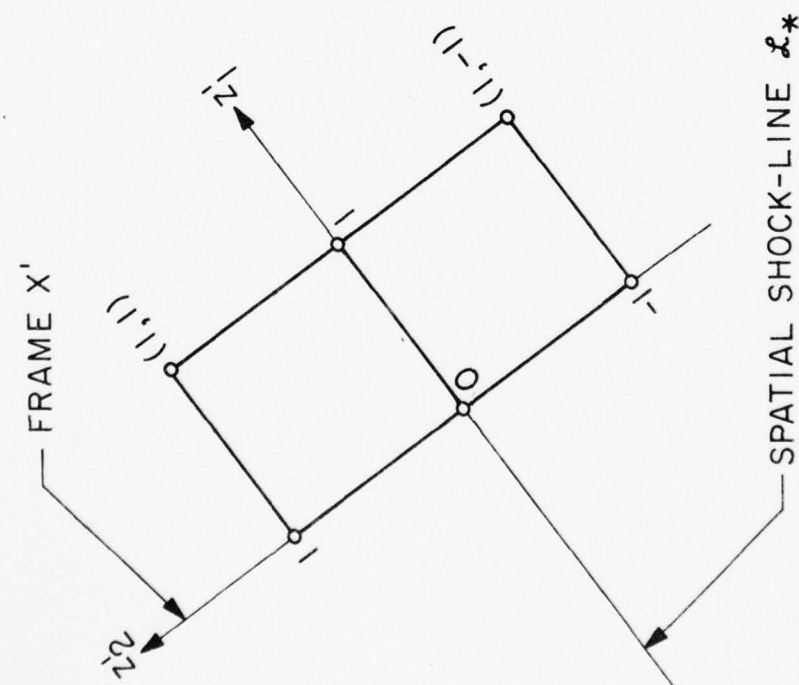FIGURE 4. MATERIAL 2. PLANE-STRAIN DOMAIN OF ELLIPTICITY AND SPECIAL DEFORMATION PATHS

(a) UNDEFORMED BODY

(b) DEFORMED BODY
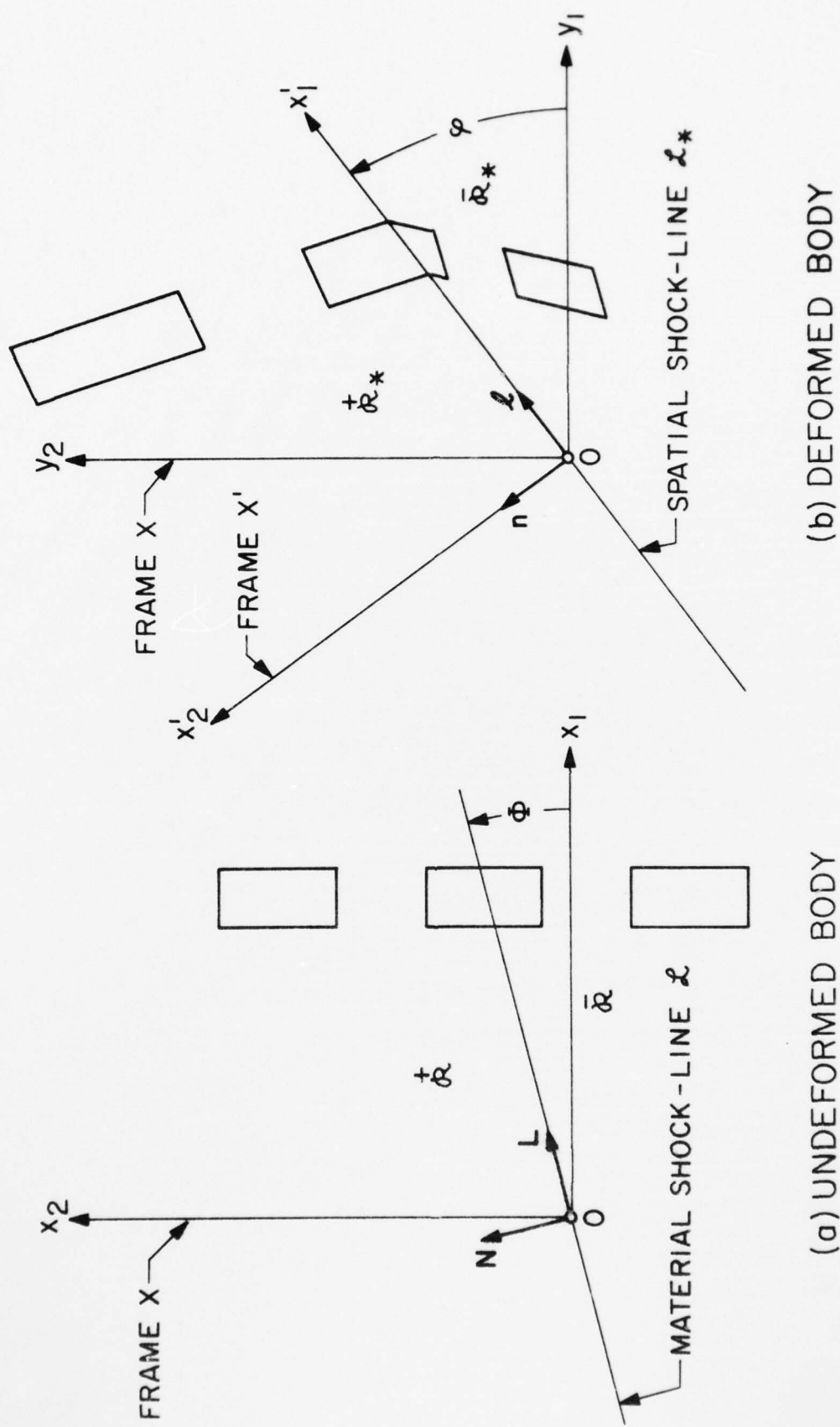
FIGURE 5. KINEMATICS OF PIECEWISE HOMOGENEOUS ELASTOSTATIC SHOCKS

FRAME X'

FRAME X'

(a) INTERMEDIATE CONFIGURATION

(b) FINAL CONFIGURATION

SPATIAL SHOCK-LINE $\mathscr{L}_*$

FIGURE 6. DECOMPOSITION OF SUPPLEMENTARY DEFORMATION $[\kappa > 0, \delta < 1]$
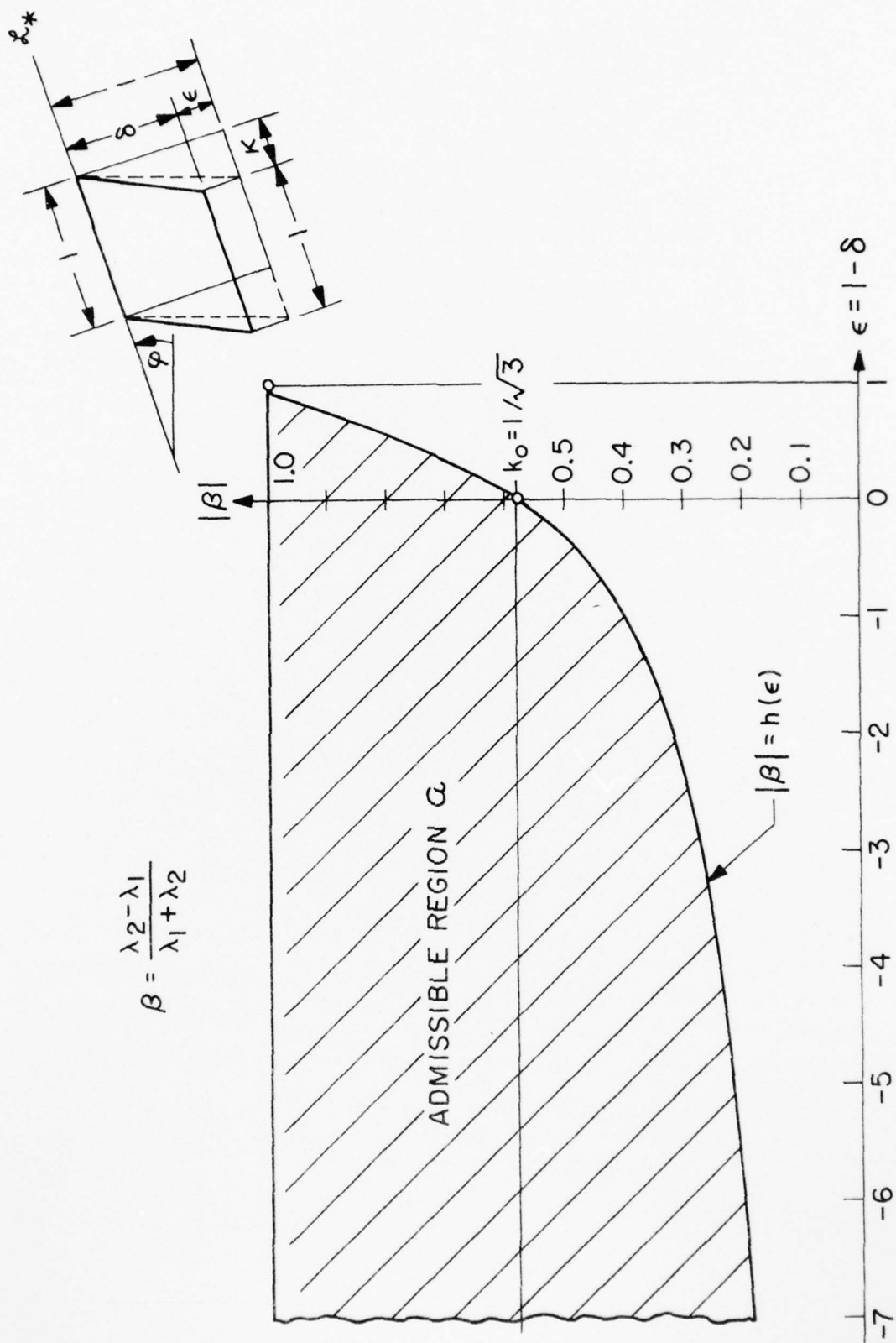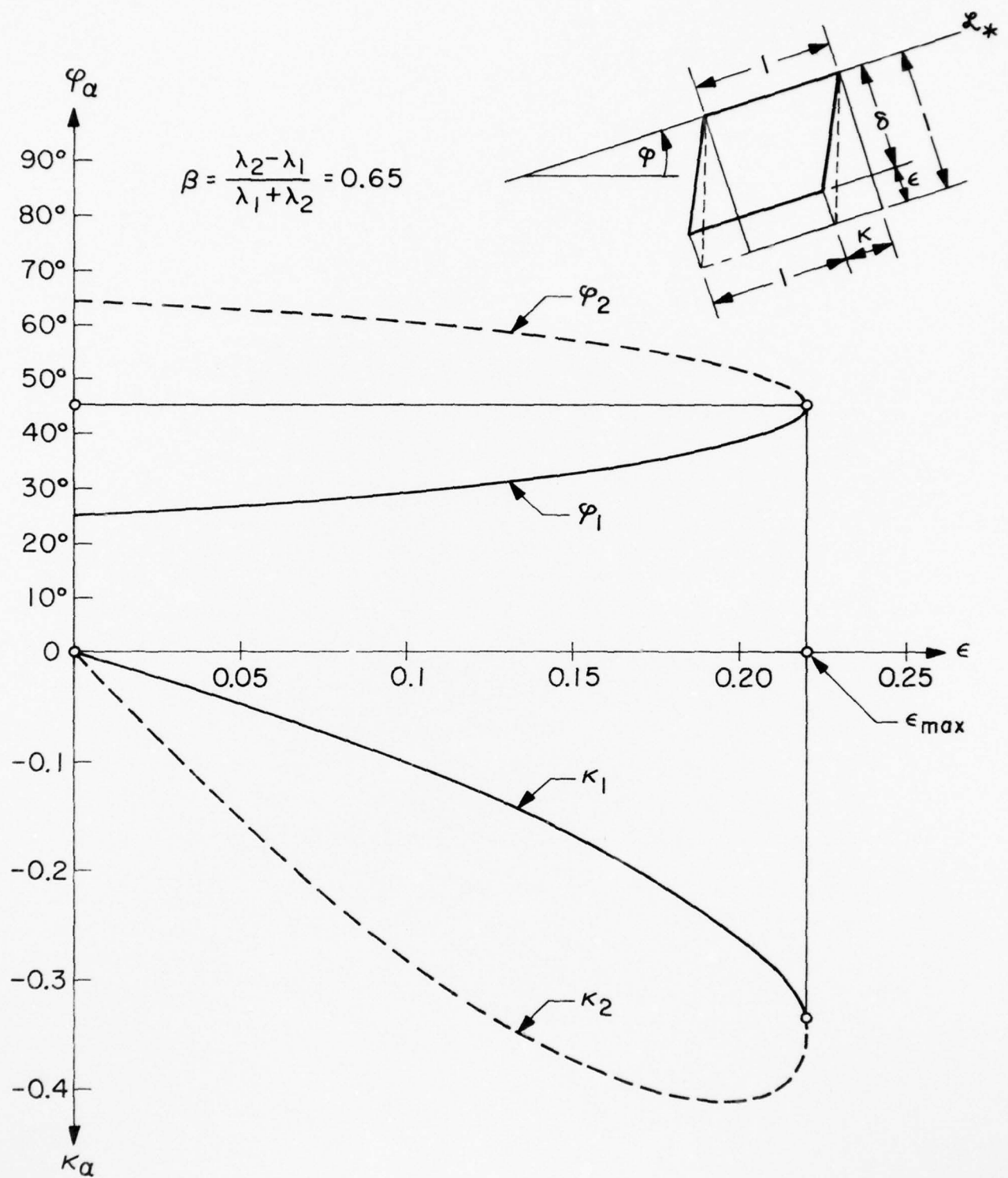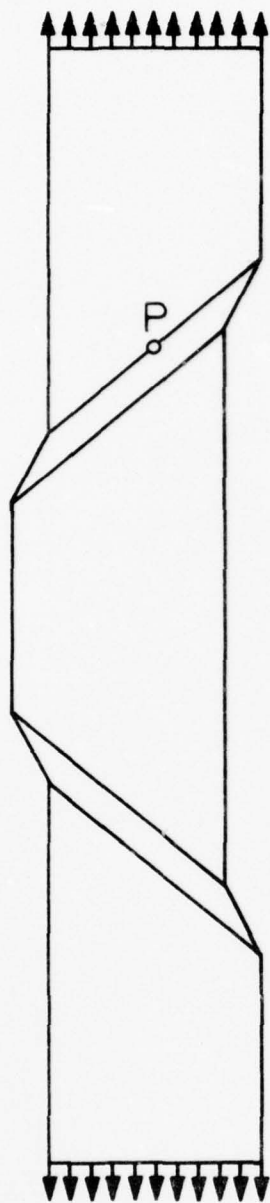
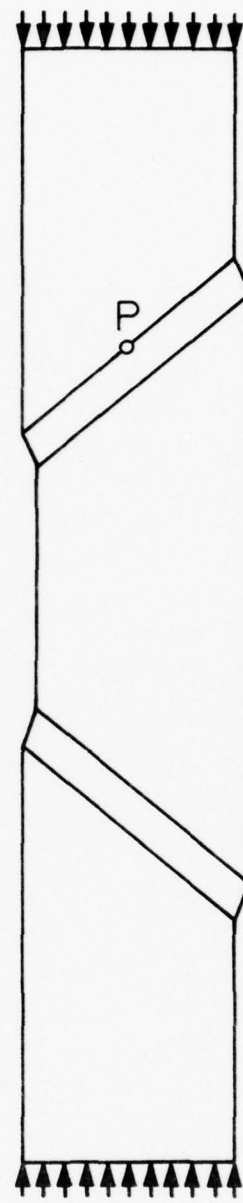FIGURE 7. MATERIAL 1. ADMISSIBLE VALUES OF PARAMETERS ϵ AND β

FIGURE 8. MATERIAL I: SHOCK-ANGLE AND AMOUNT OF SHEAR
AS A FUNCTION OF SHOCK-STRENGTH

FIGURE 9. AUXILIARIES FOR ENERGY IDENTITY

(a) TENSION   (b) COMPRESSION

FIGURE 10. SCHEMATIC REPRESENTATION OF LÜDERS LINES

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*	2a. REPORT SECURITY CLASSIFICATION
California Institute of Technology Pasadena, California 91125	2b. GROUP

3. REPORT TITLE

On the failure of ellipticity and the emergence of discontinuous deformation gradients in plane finite elastostatics.

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Research Report.

5. AUTHOR(S) *(First name, middle initial, last name)*

James K. Knowles and Eli Sternberg

TR-37

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1977	87	13

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
N00014-75-C-0196	
b. PROJECT NO.	No. 37
c. Jun 79	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research, Washington, D.C.

13. ABSTRACT

This investigation concerns equilibrium fields with discontinuous displacement gradients, but continuous displacements, in the theory of finite plane deformations of possibly anisotropic, compressible elastic solids. "Elastostatic shocks" of this kind, which resemble in many respects gas-dynamical shocks associated with steady flows, are shown to exist only if and when the governing field equations of equilibrium suffer a loss of ellipticity. The local structure of such shocks, near a point on the shock-line, is studied with particular attention to weak shocks, and an example pertaining to a shock of finite strength is explored in detail. Also, necessary and sufficient conditions for the "dissipativity" of time-dependent equilibrium shocks are established. Finally, the relevance of the analysis carried out here to localized shear failures — such as those involved in the formation of Lüders bands — is discussed.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Nonlinear elastostatics, large deformations, loss of ellipticity, equilibrium shocks, Lüders bands.						